arXiv:hep-ph/9404296v1 19 Apr 1994

.0.0

CERN-TH.7225/94

# HEAVY QUARK MASSES, MIXING ANGLES, AND SPIN-FLAVOUR SYMMETRY

MATTHIAS NEUBERT*

*Theory Division, CERN*
*CH-1211 Geneva 23, Switzerland*

ABSTRACT

In a series of three lectures, I review the theory and phenomenology of heavy quark masses and mixing angles and the status of their determination. In addition, I give an introduction to heavy quark symmetry, with the main emphasis on the development of heavy quark effective field theory and its application to obtain a model-independent determination of $|V_{cb}|$.

(Lectures presented at TASI-93, Boulder, Colorado, 1993)

CERN-TH.7225/94
April 1994

*Address before Oct. 1993: Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309

## Introduction

The rich phenomenology of weak decays has always been a source of information about the structure of elementary particle interactions. A long time ago, $\beta$- and $\mu$-decay experiments revealed the nature of the effective flavour-changing interactions at low momentum transfer. Today, we are in a similar situation: Weak decays of hadrons containing heavy quarks are employed for tests of the standard model and measurements of its parameters. In particular, they offer the most direct way to determine the weak mixing angles and to test the unitarity of the Kobayashi–Maskawa matrix. On the other hand, hadronic weak decays also serve as a probe of that part of strong-interaction phenomenology which is least understood: the confinement of quarks and gluons into hadrons. In fact, it is this intricate interplay between weak and strong interactions that makes this field challenging and attractive to many theorists.

Over the last decade or so, a lot of information has been collected about heavy quark decays from experiments on the $\Upsilon(4s)$ resonance, and more recently at $e^+e^-$ and hadron colliders. This has led to a rather detailed knowledge of the flavour sector of the standard model and many of the parameters associated with it. There have been several great discoveries in this field, such as $B_d$–$\bar{B}_d$ mixing[1,2], $b \to u$ transitions[3–6], and rare decays induced by penguin operators[7]. Yet there is much more to come. Hopefully, the approval of the first $B$-meson factory at SLAC has opened the way for a bright future for $B$-physics. At the same time, upgrades of the existing facilities at Cornell, Fermilab, and LEP will provide a wealth of data within the coming years.

However, my main purpose here is to talk about theory, and fortunately there has been a lot of progress and enthusiasm in this field in recent years. This is related to the discovery of heavy quark symmetry[8–13] and the development of the heavy quark effective theory[14–24], which is a low-energy effective theory that describes the strong interactions of a heavy quark with light quarks and gluons. The excitement about these developments is caused by the fact that they allow (some) model-independent predictions in an area in which "progress" in theory often meant nothing more than the construction of a new model, which could be used to estimate some strong-interaction hadronic matrix elements. Therefore, I hope that you do not mind it if the part about the heavy quark effective theory constitutes the main body of these notes; this is where the main progress has been achieved from the theoretical point of view. Also, I have recently finished a long review article on heavy quark symmetry[25], which clearly simplifies the task of writing up lecture notes.

In the first lecture, I will discuss the theory of heavy quark masses, their definition in perturbation theory, and their determination from QCD sum rules. The second lecture provides an introduction to the standard model description of quark mixing, different parametrizations of the Kobayashi–Maskawa matrix, the status of the determination of the mixing angles, and the physics of the unitarity triangle. The third lecture, which covers more than half of these notes, is devoted to a review of the ideas on heavy quark symmetry and the formalism of the heavy quark

effective theory. In particular, I will discuss the theory of the model-independent determination of $|V_{cb}|$ from exclusive semileptonic $B \to D^* \ell\, \bar\nu$ decays. At the end of each lecture you will find some suggestions for little exercises, which are typically related to the derivation of important equations given in the notes. I have tried to select problems that are fun to solve and contain some interesting pieces of physics. You are invited to see if I am right.

## 1. Heavy Quark Masses

Because of confinement at large distance scales, quarks and gluons do not appear among the physical states of QCD. There is thus no natural, physical definition of quark masses. Rather, several definitions are possible and have been proposed, and it is often a matter of convenience which one to use. Most of these definitions are tied to the framework of perturbation theory. In this lecture, I will discuss some of the most common mass definitions and their interrelation. Special emphasis is put on the discussion of the running quark mass in the context of the renormalization group. I will then discuss how values for the bottom and charm quark masses are obtained from QCD sum rules. Very briefly, I will touch upon the subtle issue of an infrared renomalon in the pole mass, which implies an intrinsic uncertainty in these determinations.

### *1.1. Quark Mass Definitions*

For heavy quarks, the nonrelativistic bound-state picture suggests the notion of a pole mass $m_Q^{\rm pole}$ defined, order by order in perturbation theory, as the position of the singularity in the renormalized quark propagator. One can show that the pole mass is gauge-invariant, infrared finite, and renormalization-scheme independent[26]. It is thus a meaningful "physical" parameter as long as the heavy quark is not exactly on-shell. For instance, the pole mass appears in the formula for the energy levels in quarkonium systems.

An alternative gauge-invariant definition is provided by the running quark mass in some subtraction scheme. One usually works with the modified minimal subtraction ($\overline{\rm MS}$) scheme[27] and denotes the running mass by $\overline{m}_Q(\mu)$. Running quark masses are used when there is a large momentum scale $\mu \gg m_Q$ in the problem, so as to absorb large logarithms, which would otherwise render perturbation theory invalid. Of course, one can use other subtraction schemes such as minimal subtraction[28] (MS). Sometimes, it may even be convenient to use a gauge-dependent definition of a heavy quark mass such as the so-called Euclidean mass $m_Q^{\rm Eucl}$, which is defined by a subtraction at the Euclidean point[29] $p^2 = -m_Q^2$. What is important is that these perturbative definitions are related to each other in a calculable way, order by order in perturbation theory.

One may also think of obvious nonperturbative definitions of a heavy quark mass. For instance, one could define $m_Q$ to be one half of the mass of the ground state in the corresponding $(Q\bar{Q})$ quarkonium system, or as the mass of the lightest hadron containing the heavy quark, or as that mass minus some fixed binding energy, etc. The problem is, of course, that these definitions are ad hoc, and relating them to each other or to the perturbative definitions given above would require to solve QCD, a task that is presently beyond our calculational skills. There is, however, a bridge between the two classes of definitions, which is provided by the heavy quark effective theory, which is an effective field theory appropriate to describe the soft interactions of an almost on-shell heavy quark with light quarks and gluons. There,

one can define a parameter $\bar{\Lambda}$, which corresponds to the effective mass of the light degrees of freedom in a hadron $H_Q$ containing a single heavy quark $Q$, in terms of a gauge-invariant, infrared finite, and renormalization-scheme independent hadronic matrix element[30]. One can then define a heavy quark mass $m_Q^*$ as[31] $m_Q^* = m_{H_Q} - \bar{\Lambda}$. In perturbation theory, and up to corrections of order $1/m_Q$, the mass $m_Q^*$ coincides with the pole mass. However, the above definition is more general, as it does not rely on a perturbative expansion.

*1.2. Quark Masses in Perturbation Theory*

Let me now discuss the concept of heavy quark masses to one-loop order in perturbation theory. The bare quark mass $m_Q^{\rm bare}$ appearing in the QCD Lagrangian is related to the renormalized mass $m_Q^{\rm ren}$ by a counter term,

$$m_Q^{\rm bare} = m_Q^{\rm ren} - \delta m_Q \,, \tag{1}$$

where $m_Q^{\rm bare}$ and $\delta m_Q$ are divergent quantities. The counter term is chosen such that the renormalized mass is finite. Its value depends, however, on the subtraction prescription. In the $\overline{\rm MS}$ scheme, the renormalized mass will depend on some subtraction scale $\mu$, i.e. $m_Q^{\rm ren} = \overline{m}_Q(\mu)$. To calculate $\delta m_Q$ in perturbation theory, one has to evaluate the heavy quark self-energy $\Sigma(p)$ shown in Fig. 1. The relation is

$$\delta m_Q = \text{divergent part of } \Sigma(\not{p} = m_Q) + \text{scheme-dependent finite terms.} \tag{2}$$

To calculate the self-energy at one-loop order, it is convenient to use dimensional regularization[32,33], i.e. to work in $D = 4 - 2\epsilon$ space-time dimensions. Then the renormalized coupling constant is related to the bare one by[34]

$$\alpha_s^{\rm bare} = Z_\alpha \, \alpha_s(\mu) \, \mu^{2\epsilon} = \alpha_s \, \mu^{2\epsilon} + O(\alpha_s^2) \,. \tag{3}$$

From a straightforward calculation, one obtains at one-loop order

$$\begin{aligned} \Sigma(\not{p} = m_Q) &= m_Q \, \frac{\alpha_s}{3\pi} \left( \frac{m_Q^2}{4\pi\mu^2} \right)^{-\epsilon} \Gamma(\epsilon) \, \frac{3-2\epsilon}{1-2\epsilon} \,, \\ &= m_Q \, \frac{\alpha_s}{\pi} \left( \frac{1}{\hat{\epsilon}} + \ln \frac{\mu^2}{m_Q^2} + \frac{4}{3} \right) + O(\epsilon) \,, \end{aligned} \tag{4}$$

where $1/\hat{\epsilon} = 1/\epsilon - \gamma + \ln 4\pi$. Notice that this result is still $\mu$-independent, since the $\mu$ dependence of $\alpha_s/\hat{\epsilon}$ cancels the $\mu$-dependent logarithm. A scale dependence appears only when one subtracts the ultraviolet divergence. In the $\overline{\rm MS}$ scheme, the mass counter term is defined to subtract the $1/\hat{\epsilon}$-pole in the self-energy, i.e.

$$\begin{aligned} \delta m_Q^{\overline{\rm MS}} &= m_Q \, \frac{\alpha_s}{\pi\hat{\epsilon}} \,, \\ \overline{m}_Q(\mu) &= m_Q \left( 1 + \frac{\alpha_s}{\pi\hat{\epsilon}} \right) . \end{aligned} \tag{5}$$

The pole mass, on the other hand, is defined so as to absorb the entire self-energy on-shell. Hence

$$\begin{aligned} \delta m_Q^{\rm pole} &= \Sigma(\not{p} = m_Q)\,, \\ m_Q^{\rm pole} &= m_Q \left\{1 + \frac{\alpha_s}{\pi}\left(\frac{1}{\hat\epsilon} + \ln\frac{\mu^2}{m_Q^2} + \frac{4}{3}\right)\right\}. \end{aligned} \tag{6}$$

Comparing the two results, one obtains

$$\begin{aligned} \overline{m}_Q(\overline{m}_Q) &= m_Q^{\rm pole}\left\{1 - \frac{4\alpha_s(\overline{m}_Q)}{3\pi}\right\}, \\ \overline{m}_Q(\mu) &= \overline{m}_Q(\overline{m}_Q)\left\{1 - \frac{\alpha_s}{\pi}\,\ln\frac{\mu^2}{\overline{m}_Q^2}\right\}. \end{aligned} \tag{7}$$

The first equation relates different perturbative definitions of the heavy quark mass, namely the pole mass to the running mass in the $\overline{\rm MS}$ scheme evaluated at the scale $\mu = \overline{m}_Q$. I have written this relation as a perturbative expansion in powers of $\alpha_s(\overline{m}_Q)$. This expansion is also known to two-loop order from a calculation by Gray et al.[35] Similar relations exist in other subtraction schemes, e.g.

$$\begin{aligned} m_Q^{\rm MS}(m_Q^{\rm MS}) &= m_Q^{\rm pole}\left\{1 - \frac{\alpha_s(m_Q^{\rm MS})}{\pi}\left(\frac{4}{3} - \gamma + \ln 4\pi\right)\right\}, \\ m_Q^{\rm Eucl}(m_Q^{\rm Eucl}) &= m_Q^{\rm pole}\left\{1 - \frac{\alpha_s(m_Q^{\rm Eucl})}{\pi}\,2\ln 2\right\}, \end{aligned} \tag{8}$$

where the gauge-dependent Euclidean mass is defined in the Landau gauge[29].

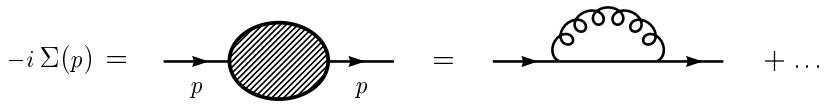


Figure 1: The quark self-energy to one-loop order in QCD.

The second relation in (7) determines the running of the heavy quark mass. In QCD, quarks become "lighter" at high-energy scales. The simple one-loop calculation presented above is, however, not sufficient to scale $\overline{m}_Q(\mu)$ up to very large scales. For instance, using (7) to extrapolate the mass of the bottom quark up to a typical grand unification scale $M_{\rm GUT} \sim 10^{16}$ GeV, one would obtain a meaningless result:

$$\frac{\overline{m}_b(M_{\rm GUT})}{\overline{m}_b(\overline{m}_b)} = 1 - \frac{\alpha_s}{\pi}\,\ln\frac{M_{\rm GUT}^2}{\overline{m}_b^2} \simeq 1 - 71\,\frac{\alpha_s}{\pi} = ? \tag{9}$$

Of course, the problem of large logarithms in perturbation theory is not specific to this case, but is encountered frequently. Large logarithms can be controlled in a

systematic way using the beautiful machinery of the renormalization group. It is worth going through the solution in great detail.

*1.3. Renormalization-Group Improvement*

For the purpose of this section, I will switch to a slightly more general notation and rewrite (7) as

$$m(\mu) = m(\mu_0) \left\{ 1 - \frac{\alpha_s}{\pi} \ln \frac{\mu^2}{\mu_0^2} + O(\alpha_s^2) \right\}. \tag{10}$$

If $\mu \gg \mu_0$, the logarithm can be so large that $\alpha_s \ln \mu^2/\mu_0^2$ becomes of order unity, and ordinary perturbation theory breaks down. In fact, one can show that at any given order in perturbation theory there will be large logarithms of the type

$$\left( \alpha_s \ln \frac{\mu^2}{\mu_0^2} \right)^n \sim \left( \frac{\ln \mu^2/\mu_0^2}{\ln \mu^2/\Lambda^2_{\rm QCD}} \right)^n , \tag{11}$$

where I have used the fact that the running coupling constant scales like $\alpha_s(\mu) \sim 1/\ln(\mu^2/\Lambda^2_{\rm QCD})$. It is necessary to resum these "leading logarithms" to all orders in perturbation theory. This is achieved by solving the renormalization-group equation (RGE) for the running quark mass,

$$\mu \frac{\rm d}{{\rm d}\mu} m(\mu) = \gamma(\alpha_s)\, m(\mu) . \tag{12}$$

The anomalous dimension $\gamma$ has a perturbative expansion in the renormalized coupling constant:

$$\gamma(\alpha_s) = \gamma_0 \frac{\alpha_s}{4\pi} + \gamma_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \dots . \tag{13}$$

The coefficients $\gamma_i$ are known to three-loop order[36,37]. For our purposes, however, it is sufficient to note that[26]

$$\gamma_0 = -8 , \qquad \gamma_1 = -\frac{404}{3} + \frac{40}{9} n_f , \tag{14}$$

where $n_f$ denotes the number of quark flavours with mass below $\mu$. Throughout these notes I will evaluate the QCD coefficients for $N_c = 3$ colours, and only display the dependence on the number of flavours explicitly. The value of $\gamma_0$ follows from the one-loop result (10).

The next step is to rewrite the RGE in the form of a partial differential equation, making explicit the scale dependence of the renormalized coupling constant. This gives

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s(\mu)} - \gamma(\alpha_s) \right) m(\mu) = 0 . \tag{15}$$

The $\beta$-function

$$\beta(\alpha_s) = \mu \frac{\partial \alpha_s(\mu)}{\partial \mu} = -2\alpha_s \left[ \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \dots \right] \tag{16}$$

describes the running of the coupling constant. The one- and two-loop coefficients are[38–42]

$$\beta_0 = 11 - \frac{2}{3} n_f \,, \qquad \beta_1 = 102 - \frac{38}{3} n_f \,. \tag{17}$$

The exact solution of the RGE can be written in the form

$$m(\mu) = U(\mu, \mu_0)\, m(\mu_0) \,, \tag{18}$$

with the evolution operator[27,43,44]

$$U(\mu, \mu_0) = \exp \int\limits_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \mathrm{d}\alpha \, \frac{\gamma(\alpha)}{\beta(\alpha)} \,. \tag{19}$$

The trick is to obtain a perturbative expansion in the exponent of this expression by inserting the expansions for the anomalous dimension and $\beta$-function. After a simple calculation, one finds

$$U(\mu, \mu_0) = \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{\gamma_0/2\beta_0} \left\{ 1 + \frac{\alpha_s(\mu_0) - \alpha_s(\mu)}{4\pi} \, \frac{\gamma_1 \beta_0 - \beta_1 \gamma_0}{2\beta_0^2} + O(\alpha_s^2) \right\} . \tag{20}$$

In this result, the running coupling constant has two-loop accuracy. The corresponding expression is

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)} \left[ 1 - \frac{\beta_1}{\beta_0^2} \, \frac{\ln\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)}{\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)} \right] , \tag{21}$$

where $\Lambda_{\mathrm{QCD}}$ is a scheme-dependent scale parameter[†] The factor containing the ratio of the running coupling constants in (20) sums the leading logarithms (11) to all orders in perturbation theory. Keeping just this factor corresponds to the so-called leading logarithmic approximation (LLA), which is often used to attack problems containing widely separated scales. Note that to calculate this factor, one only needs to compute the one-loop coefficient of the anomalous dimension. This is a rather trivial task, as it suffices to calculate the $1/\epsilon$-pole in the quark self-energy. Once the leading scaling behaviour has been factored out, perturbation theory becomes well-behaved, i.e. the next-to-leading corrections in the parenthesis in (20) obey a perturbative expansion free of large logarithms. This is why the approach is

[†]The value of $\alpha_s(\mu_0)$ at some reference scale $\mu_0$ can be used to eliminate the dependence on $\Lambda_{\mathrm{QCD}}$. Nowadays, it is convenient to choose $\mu_0 = m_Z$, since $\alpha_s(m_Z)$ is known with high accuracy.

called "renormalization-group improved perturbation theory". The terms of order $\alpha_s$, which I have shown explicitly, correspond to the next-to-leading logarithmic approximation (NLLA). Including them, one sums logarithms of the type

$$\alpha_s \left(\alpha_s \, \ln \frac{\mu^2}{\mu_0^2}\right)^n \tag{22}$$

to all orders. To achieve such an accuracy, it is necessary to calculate the two-loop coefficients of the anomalous dimension and $\beta$-function.

Combining the above results and setting $\mu_0 = m_Q$, one obtains the running quark mass to next-to-leading order in renormalization-group improved perturbation theory. The result is

$$m_Q(\mu) = m_Q(m_Q) \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{4/\beta_0} \left\{1 + S \, \frac{\alpha_s(m_Q) - \alpha_s(\mu)}{\pi} + O(\alpha_s^2)\right\}, \tag{23}$$

where

$$S = -\frac{5}{6} + \frac{34}{3\beta_0} - \frac{107}{\beta_0^2} \, . \tag{24}$$

It is to be understood that the number of flavours changes as $\mu$ crosses a quark threshold[45]. For instance, when one uses (23) to scale the running bottom quark mass up to large scales, $n_f$ changes from 5 to 6 when $\mu$ crosses the top quark mass. At the same time, the QCD scale parameter $\Lambda_{\rm QCD}$ in expression (21) for the running coupling constant changes, so that $\alpha_s(\mu)$ is a continuous function. From the point of view of convergence of perturbation theory, eq. (23) can be used to evaluate the running mass at an arbitrarily large scale $\mu$. However, I have derived this result assuming that there are only QCD interactions. But at high-energy scales, other gauge and Yukawa interactions become important as well. They are most significant for the running of the top quark mass, as I will discuss in detail in the next section. In the case of the bottom quark, the result that renormalization effects lower the running mass at high-energy scales remains true. In fact, in many extensions of the standard model it is possible to obtain a unification of the bottom quark and the tau lepton masses at a typical grand unification scale:

$$m_b(M_{\rm GUT}) = m_\tau(M_{\rm GUT}) \quad \text{for} \quad M_{\rm GUT} \sim 10^{16} \text{ GeV} \, . \tag{25}$$

For more details on this, I refer to the lectures by L. Hall in this volume.

### *1.4. Running Top Quark Mass*

The interplay of gauge and Yukawa interactions leads to very interesting effects on the evolution of the top quark mass[46,47]. In my discussion below, I will focus on the standard model. The analysis for extensions of the standard model such as supersymmetry proceeds in a similar way[48–50].

Let me write the running top quark mass in terms of the Yukawa coupling $\lambda_t(\mu)$:

$$m_t(\mu) = \lambda_t(\mu)\,\frac{v(\mu)}{\sqrt{2}}\,, \tag{26}$$

where $v$ denotes the vacuum expectation value of the Higgs field, normalized so that $v(m_W) \simeq 246$ GeV. It is convenient to define a running coupling constant $\alpha_t(\mu)$ as

$$\alpha_t(\mu) = \frac{\lambda_t^2(\mu)}{4\pi}\,. \tag{27}$$

The most important contributions to the evolution of the running mass $m_t(\mu)$ come from the QCD interactions as well as from the large top Yukawa coupling. The relevant one-loop diagrams are depicted in Fig. 2. They lead to the RGE

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,\ln m_t(\mu) = -8\,\frac{\alpha_s(\mu)}{4\pi} + \frac{3}{2}\,\frac{\alpha_t(\mu)}{4\pi} + \dots\,, \tag{28}$$

where the ellipses represent other contributions, which I will neglect. Combining this with the RGE for $v(\mu)$,

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,\ln v(\mu) = -3\,\frac{\alpha_t(\mu)}{4\pi} + \dots\,, \tag{29}$$

one obtains

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,\ln \alpha_t(\mu) = -16\,\frac{\alpha_s(\mu)}{4\pi} + 9\,\frac{\alpha_t(\mu)}{4\pi} + \dots\,. \tag{30}$$

To get an idea of the structure of this equation, suppose first that the QCD coupling constant is not running, i.e. $\alpha_s = \text{const}$. Then the RGE has an infrared-stable fixed point at $\alpha_t = \frac{16}{9}\,\alpha_s$, meaning that irrespective of the value of $\alpha_t(\mu)$ at large scales, the Yukawa coupling is attracted into the fixed point as $\mu$ decreases. This is illustrated in Fig. 3. The resulting fixed-point value of the top quark mass, $m_t = \frac{4}{3}\sqrt{2\pi\alpha_s}\,v$, is entirely determined by the low-energy group structure of the theory, which is responsible for the one-loop coefficients in (30).

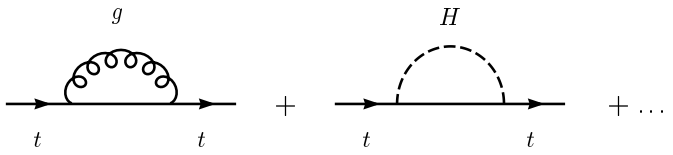


Figure 2: Dominant one-loop contributions to the self-energy of the top quark in the standard model.

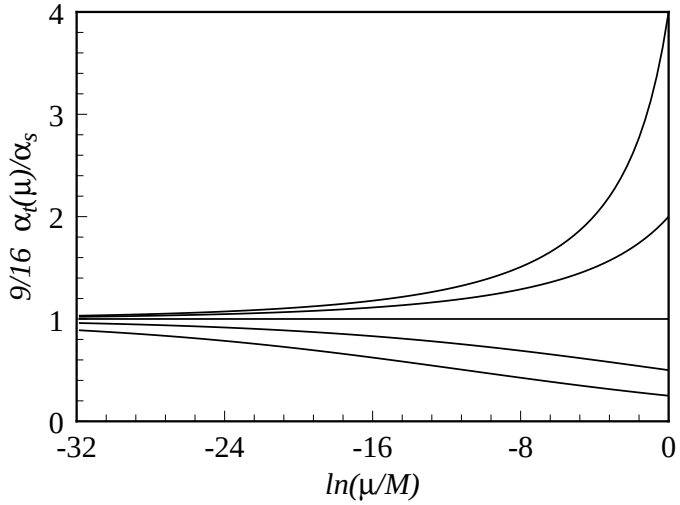


Figure 3: The fixed-point structure of the RGE (30) in the case $\alpha_s = \text{const.}$, with $4\alpha_s/\pi = 0.1$.

For running $\alpha_s(\mu)$, one has to find the simultaneous solution of (30) and the evolution equation for the strong coupling constant,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \alpha_s(\mu) = -14 \frac{\alpha_s(\mu)}{4\pi} . \tag{31}$$

Depending on the initial conditions, one finds that there are still "quasi-fixed point" solutions, where the running of $\alpha_t(\mu)$ is connected to the running of $\alpha_s(\mu)$, but independent of the value of the Yukawa coupling at large scales. In leading logarithmic approximation, the exact solution is not too hard to obtain. It reads

$$\frac{\alpha_t(\mu)}{\alpha_t(M)} = \left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right)^{8/7} \left\{1 + \frac{9}{2} \frac{\alpha_t(M)}{\alpha_s(M)} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right)^{1/7} - 1\right]\right\}^{-1} , \tag{32}$$

where $M \gg \mu$ denotes some large mass scale, at which the initial conditions are imposed. In order to illuminate the structure of this equation, let me distinguish three cases:

(i) $\alpha_t(M) \ll \alpha_s(M)$:
In this case, the top Yukawa coupling is weak, and (32) can be approximated by

$$\alpha_t(\mu) \simeq \alpha_t(M) \left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right)^{8/7} , \tag{33}$$

which is nothing but the standard QCD evolution [cf. (23)].

(ii) $\alpha_t(M) = \frac{2}{9}\,\alpha_s(M)$:
This case corresponds to the quasi-fixed point obtained by Pendleton and Ross[46], where (32) simplifies to

$$\alpha_t(\mu) = \frac{2}{9}\,\alpha_s(\mu)\,. \tag{34}$$

However, to obtain this solution requires a fine-tuning at the large scale $M$.

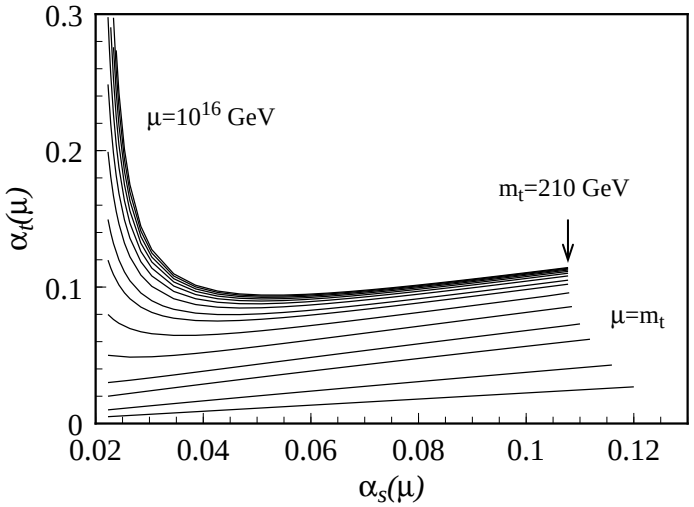


Figure 4: Evolution of the coupling constants for different initial values of $\alpha_t(M)$. I assume that $M = 10^{16}$ GeV.

(iii) $\alpha_t(M) \gg \alpha_s(M)$:
This is the most interesting case of a large top Yukawa coupling. Then the approximate solution of (32) reads

$$\alpha_t(\mu) \simeq \frac{2}{9}\,\alpha_s(\mu)\left[1 - \left(\frac{\alpha_s(M)}{\alpha_s(\mu)}\right)^{1/7}\right]^{-1}. \tag{35}$$

This is the quasi-fixed point discovered by Hill[47]. At low scales $\mu \ll M$, $\alpha_t(\mu)$ follows $\alpha_s(\mu)$ and becomes independent of the initial value $\alpha_t(M)$. The dynamics that generates the top Yukawa coupling at some large scale $M$ cannot be tested. An example of this mechanism is provided by the top condensation model of Bardeen, Hill, and Lindner[51]. Using (35), one obtains for the fixed-point value of the top quark mass

$$m_t(m_t) \simeq \frac{2v(m_t)}{3}\sqrt{\pi\,\alpha_s(m_t)}\left[1 - \left(\frac{\alpha_s(M)}{\alpha_s(m_t)}\right)^{1/7}\right]^{-1/2} \simeq 210\ \text{GeV}\,, \tag{36}$$

where I have assumed $M \simeq 10^{16}$ GeV. Again, the result is almost entirely determined by the group structure of the low-energy theory. The dependence on $M$ is very weak.

In Fig. 4, I show the evolution of the coupling constants according to (32) for a set of initial values of $\alpha_t(M)$. The quasi-fixed point behaviour is clearly visible once $\alpha_t(M)$ is large enough. A more careful analysis including all standard model contributions in the RGE gives[47] $m_t \simeq 225$ GeV (for $M \simeq 10^{16}$ GeV). Somewhat smaller values $m_t < 200$ GeV are obtained in the minimal supersymmetric standard model[48–50], depending however upon the value of $\tan\beta$.

*1.5. Determination of $m_b$ and $m_c$ from QCD Sum Rules*

In this section, I will discuss the extraction of the masses of the bottom and charm quarks from QCD sum rules, and in particular from the analysis of quarkonium spectra. The main idea of sum rules was developed in the pioneering papers of Shifman, Vainshtein, and Zakharov[29]. Their idea was to use quark–hadron duality to obtain a prediction of hadron properties from calculations in the quark–gluon theory of QCD. Consider, as an example, the vacuum polarization induced by the electromagnetic current of a bottom quark:

$$i \int \mathrm{d}^4x\, e^{iq\cdot x}\, \langle\, 0 \,|\, T\,\{j^\mu(x), j^\nu(0)\} \,|\, 0\,\rangle = (q^\mu q^\nu - g^{\mu\nu} q^2)\, \Pi(Q^2)\,, \tag{37}$$

where $j^\mu = \bar{b}\,\gamma^\mu b$, and $Q^2 = -q^2$. The Lorentz structure of the correlator follows from current conservation. The function $\Pi(Q^2)$ satisfies a once-subtracted dispersion relation

$$-\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\,\Pi(Q^2) = \frac{1}{\pi}\int \mathrm{d}s\,\frac{1}{(s+Q^2)^2}\,\mathrm{Im}\Pi(s)\,, \tag{38}$$

where by the optical theorem

$$\mathrm{Im}\Pi(s) = \frac{1}{12\pi e_b^2}\,\frac{3s}{4\pi\alpha^2}\,\sigma_s(e^+e^- \to b\,\bar{b}) \tag{39}$$

is related to a measurable cross section. Here, $e_b = -1/3$ is the electric charge of the $b$-quark, and "$b\,\bar{b}$" is a short-hand notation for "hadrons containing $b\,\bar{b}$". In the sum rule analysis one considers the moments of the correlation function, which are given by

$$M_n = \frac{1}{n!}\,\Big(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\Big)^n\,\Pi(Q^2)\Big|_{Q^2=0} = \frac{1}{\pi}\int \mathrm{d}s\, s^{-n-1}\,\mathrm{Im}\Pi(s)\,. \tag{40}$$

In principle, these moments can be extracted from experiment.

As long as $Q^2$ is not close to the resonance region, the function $\Pi(Q^2)$ can be calculated in perturbative QCD. Since $Q^2 = 0$ is far away from the physical threshold for bottomonium production, this assumption is justified in the present case. Hence, perturbative QCD should provide a good approximation to a calculation of the moments $M_n$. The leading and next-to-leading perturbative contributions to the correlator are shown in Fig. 5. They give rise to the spectral density[29]

$$\mathrm{Im}\Pi_{\mathrm{pert}}(s) = \frac{1}{4\pi}\,\frac{v(3-v^2)}{2}\,\Theta(v^2)\,\Big\{1 + \frac{4\alpha_s}{3\pi}\,\Big[\frac{\pi^2}{2v} - \frac{3+v}{4}\,\Big(\frac{\pi^2}{2} - \frac{3}{4}\Big)\Big]\Big\}\,, \tag{41}$$

where $v = \sqrt{1 - 4m_b^2/s}$ is the relative velocity of the heavy quarks. In this expression one uses the Euclidean mass for $m_b$ in order to minimize the effect of radiative corrections [see (8)]. For $s \gg 4m_b^2$, the perturbative spectral density leads to the well-known cross section

$$\sigma_s(e^+e^- \to b\,\bar{b})_{\rm pert} \stackrel{s\gg 4m_b^2}{\to} \frac{4\pi\alpha^2}{3s}\,N_c\,e_b^2\left(1+\frac{\alpha_s}{\pi}\right). \tag{42}$$

However, the above form of the spectral density includes threshold effects, which become important when $s$ comes closer to the threshold region.

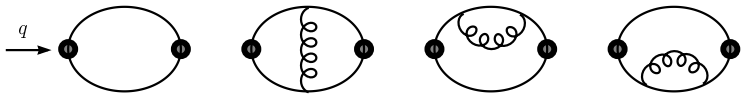


Figure 5: Perturbative contributions to the correlator $\Pi(Q^2)$. The current operators are represented by circles.

In QCD sum rules, one usually adds to the perturbative contributions nonperturbative corrections, which are power suppressed and appear at higher orders in the operator product expansion (OPE) of correlation functions such as $\Pi(Q^2)$. In the case at hand, the leading nonperturbative corrections are proportional to the gluon condensate $\langle\alpha_s G^2\rangle \sim 0.04$ GeV$^4$, which in sum rules is treated as a phenomenological parameter[29]. In the case of the bottomonium system, one is in the fortunate situation that the nonperturbative contributions to the spectral function are very small, as they are suppressed by the ratio $\langle\alpha_s G^2\rangle/m_b^4 \sim 10^{-4}$. Their effect on the mass of the bottom quark is below 1% and can safely be neglected.

The idea is now to construct two equivalent representations for the moments $M_n$, a "theoretical" one based on expression (41) for the spectral function, and a "phenomenological" one based on a measurement of the $e^+e^- \to b\,\bar{b}$ cross section:

$$\begin{aligned} M_n^{\rm th} &\simeq \frac{1}{\pi}\int\limits_{4m_b^2}^{\infty} {\rm d}s\, s^{-n-1}\,{\rm Im}\Pi_{\rm pert}(s)\,, \\ M_n^{\rm exp} &= \frac{1}{\pi}\int\limits_{M^2_{\Upsilon(1s)}}^{\infty} {\rm d}s\, s^{-n-1}\,{\rm Im}\Pi_{\rm exp}(s)\,. \end{aligned} \tag{43}$$

The moments $M_n^{\rm th}$ are very sensitive functions of $m_b$. By comparing them to a phenomenological expression that uses detailed experimental information in the $b\,\bar{b}$ vector channel, one can extract an accurate value for the bottom quark mass. Experimentally, six resonances have been identified in this channel. Their masses $M_{\rm R}$ and

electronic widths $\Gamma_{\rm R}(e^+e^-)$ are known rather precisely[52]. To evaluate the integrals over the experimental cross section, one then makes the following approximation:

$$\mathrm{Im}\Pi_{\rm exp}(s) \simeq \frac{3}{4\alpha^2 e_b^2} \sum_{\rm R} M_{\rm R}\,\Gamma_{\rm R}(e^+e^-)\,\delta(s-M_{\rm R}^2) + \mathrm{Im}\Pi_{\rm pert}(s)\,\Theta(s-s_0)\,, \tag{44}$$

where the sum is over the narrow resonances $\Upsilon(1s)$ to $\Upsilon(6s)$, and the continuum above some threshold value $s_0$ is approximated by perturbation theory. This can be justified by quark–hadron duality. Inserting this ansatz into (43) and equating the two representations for the moments, one obtains the sum rules

$$\frac{1}{\pi}\int\limits_{4m_b^2}^{s_0} \mathrm{d}s\, s^{-n-1}\,\mathrm{Im}\Pi_{\rm pert}(s) \simeq \frac{3}{4\pi\alpha^2 e_b^2} \sum_{\rm R} M_{\rm R}^{-2n-1}\,\Gamma_{\rm R}(e^+e^-)\,. \tag{45}$$

The goal is to find an optimal set of parameters $(m_b, s_0)$ such that $M_n^{\rm th} \simeq M_n^{\rm exp}$ for many values of $n$. In practice, the perturbative calculation breaks down for large values of $n$, since the Coulomb corrections proportional to $\alpha_s/v$ in (41) then become too large. Typically, one is limited to values $n < 10$. As mentioned above, the non-perturbative corrections are very small (below 1%) in the case of the bottomonium system. They become sizeable, however, in the charmonium system, where a similar analysis can be performed. Therefore, $m_b$ can be extracted with larger accuracy than $m_c$.

Let me then quote the numerical results obtained by various authors following the strategy outlined above. Values for the Euclidean mass of the bottom quark have been obtained by Shifman et al.[29] ($m_b^{\rm Eucl} = 4.23 \pm 0.05$ GeV), Guberina et al.[53] ($m_b^{\rm Eucl} = 4.19 \pm 0.06$ GeV), and Reinders[54] ($m_b^{\rm Eucl} = 4.17 \pm 0.02$ GeV). The differences in these values are mainly due to changes in the experimental input numbers. The result of Reinders is the most up-to-date one. Values for the Euclidean mass of the charm quark have been obtained many years ago by Novikov et al.[55] ($m_c^{\rm Eucl} \simeq 1.25$ GeV) and Shifman et al.[29] ($m_c^{\rm Eucl} = 1.26 \pm 0.02$ GeV). They have not changed since then. Converting these results into pole masses using (8), one finds[54]

$$\begin{aligned} m_b^{\rm pole} &= 4.55 \pm 0.05\ \text{GeV}\,, \\ m_c^{\rm pole} &= 1.45 \pm 0.05\ \text{GeV}\,. \end{aligned} \tag{46}$$

The increase in the error is due to an additional uncertainty in the value of $\alpha_s(m_Q)$.

Let me finally mention some alternative approaches to extract pole masses from QCD sum rules. Voloshin has proposed to resum the large Coulomb corrections proportional to $(\alpha_s/v)^n$ to all orders in perturbation theory, using a nonrelativistic approach[56]. This allows one to go to higher moments, thereby suppressing the resonance contributions to the sum rule. The disadvantage is that some relativistic corrections are not taken into account. The result is a rather large value of the pole mass of the bottom quark[57],

$$m_b^{\rm pole} = 4.79 \pm 0.03\ \text{GeV}\,. \tag{47}$$

Another alternative is to obtain heavy quark masses from the study of sum rules for heavy–light bound states such as the $B$- and $B^*$-mesons. From such an analysis, Narison obtains[58]

$$m_b^{\rm pole} = 4.56 \pm 0.05 \ {\rm GeV} \,. \tag{48}$$

A more recent analysis using the heavy quark expansion to order $1/m_Q$ gives[25,59,60]

$$\begin{aligned} m_b &= 4.71 \pm 0.07 \ {\rm GeV} \,, \\ m_c &= 1.37 \pm 0.12 \ {\rm GeV} \,. \end{aligned} \tag{49}$$

It should be noticed, however, that these sum rules are more sensitive to nonperturbative corrections than are the sum rules for the quarkonium systems.

### *1.6. Infrared Renomalons*

As emphasized at the beginning of this lecture, the fact that at low energies quarks are always confined into hadrons prohibits a physical, on-shell definition of quark masses. Although for heavy quarks the notion of a pole mass is widely used, such a concept becomes meaningless beyond perturbation theory. No precise definition of the pole mass can be given once nonperturbative effects are taken into account.

It is interesting that indications for an intrinsic ambiguity in the pole mass can already be found within the context of perturbation theory, when one studies the asymptotic behaviour of the perturbative series for the quark self-energy[61,62]. It can be shown that the existence of so-called infrared renomalons[63–67] generates a factorial divergence in the expansion coefficients. Roughly speaking, the reason is that self-energy corrections to the gluon propagator in the diagram depicted in Fig. 1 effectively introduce the running coupling constant $\alpha_s(\sqrt{k^2})$, where $k^2$ is the square of the virtual gluon momentum in Euclidean space. Since $\alpha_s(\sqrt{k^2})$ increases as $k^2$ decreases, small loop momenta become more important. One may reorganize the perturbative expansion as an expansion in the small coupling constant $\alpha_s(m_Q)$ using

$$\alpha_s(\sqrt{k^2}) \simeq \alpha_s(m_Q) \sum_{n=0}^{\infty} \left( \frac{\beta_0}{4\pi} \, \alpha_s(m_Q) \, \ln \frac{m_Q^2}{k^2} \right)^n . \tag{50}$$

When one then performs the loop integral over $k^2$, the extra logarithms give rise to a growth of the expansion coefficients proportional to $n!$. In such a situation, it is in principle not possible to improve the accuracy of perturbation theory by including more and more terms in the perturbative series. Starting from some order $n$, the size of the corrections increases. The best that can be achieved is to truncate the series at an optimal value of $n$. This introduces an intrinsic error, which depends exponentially on $1/\alpha_s(m_Q)$. In the case of the pole mass, one can show that the irreducible uncertainty in the value of $m_Q^{\rm pole}$ is of order[61,62]

$$\Delta m_Q^{\rm pole} = \frac{8}{3\beta_0} \, m_Q \, \exp\left( - \frac{2\pi}{\beta_0 \, \alpha_s(m_Q)} \right) \simeq \frac{8 \Lambda_{\rm QCD}}{3\beta_0} \,. \tag{51}$$

It is thus of the order of the confinement scale $\Lambda_{\rm QCD}$. The fact that $\Delta m_Q^{\rm pole}/m_Q^{\rm pole}$ vanishes in the $m_Q \to \infty$ limit justifies the concept of a pole mass for heavy quarks, at least in an approximate sense that is limited to the context of perturbation theory.

Although one has to be careful when using (51) to obtain an estimate of the intrinsic uncertainty in the pole mass, I will insert $\Lambda_{\rm QCD} \sim 150$ MeV to find $\Delta m_Q^{\rm pole} \sim 50$ MeV. This uncertainty should be kept in mind when considering numerical results for heavy quark masses obtained, for instance, using QCD sum rules.

### *1.7. Exercises*

- Derive the one-loop expression (4) for the on-shell quark self-energy in QCD. The necessary loop integrals in $D$ space-time dimensions can be found in any reasonable textbook on quantum field theory.
- Show that (19) is the exact solution of the RGE (15), and derive the next-to-leading logarithmic approximation (20) for the evolution operator.
- By calculating the $1/\epsilon$-poles of the diagrams shown in Fig. 2, derive the one-loop RGE (28) for the running top quark mass.
- Solve the RGE (30) for the case $\alpha_s = {\rm const.}$, and show that there is an infrared-stable fixed point at $\alpha_t = \frac{16}{9}\,\alpha_s$.
- Derive eqs. (32)–(35).

## 2. Quark Mixing in the Standard Model

In this lecture, I give an introduction to flavour-changing decays and quark mixing in the standard model. I will discuss quark mixing in the cases of two and three generations, introduce some useful parametrizations of the Kobayashi–Maskawa matrix, and briefly review the status of the direct experimental determination of the entries in this matrix from tree-level processes. I will then turn to the geometrical interpretation of the mixing matrix provided by the unitarity triangle, and finish with some remarks on rare decays. Since you probably have heard many lectures on these subjects, my presentation will be rather short. Let me refer those of you who want more details to two excellent review articles that I like very much: *A Top Quark Story* by Buras and Harlander[68], and *CP Violation* by Nir[69].

### *2.1. Cabibbo–Kobayashi–Maskawa Matrix*

Let me briefly remind you of some facts about the flavour sector of the standard model. Below mass scales of order $m_W \sim 80$ GeV, the standard model gauge group $\mathrm{SU_C}(3) \times \mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ is spontaneously broken to $\mathrm{SU_C}(3) \times \mathrm{U_{em}}(1)$, since the scalar Higgs doublet $\phi$ acquires a vacuum expectation value

$$\langle \phi \rangle = \left\langle \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \right\rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad v \simeq 246 \text{ GeV}. \tag{52}$$

This gives masses to the $W$- and $Z$-bosons, as well as to the quarks and leptons. The quark masses arise from the quark Yukawa couplings to the Higgs doublet, which in the unbroken theory are assumed to be of the most general form that is invariant under local gauge transformations. The Yukawa interactions are written in terms of the weak eigenstates $q'$ of the quark fields, which have definite transformation properties under $\mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$. After the symmetry breaking, one redefines the quark fields so as to obtain the mass terms in the canonical form. This has an interesting effect on the form of the flavour-changing charged-current interactions. In the weak basis, these interactions have the form

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \left(\bar{u}'_{\mathrm{L}}, \bar{c}'_{\mathrm{L}}, \bar{t}'_{\mathrm{L}}\right) \gamma^\mu \begin{pmatrix} d'_{\mathrm{L}} \\ s'_{\mathrm{L}} \\ b'_{\mathrm{L}} \end{pmatrix} W^\dagger_\mu + \text{h.c.} \tag{53}$$

In terms of the mass eigenstates $q$, however, this becomes

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \left(\bar{u}_{\mathrm{L}}, \bar{c}_{\mathrm{L}}, \bar{t}_{\mathrm{L}}\right) \gamma^\mu \, V_{\mathrm{KM}} \begin{pmatrix} d_{\mathrm{L}} \\ s_{\mathrm{L}} \\ b_{\mathrm{L}} \end{pmatrix} W^\dagger_\mu + \text{h.c.} \tag{54}$$

The Kobayashi–Maskawa mixing matrix

$$V_{\mathrm{KM}} \simeq \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{55}$$

is a unitary matrix in flavour space. In the general case of $n$ quark generations, $V_{\rm KM}$ would be an $n \times n$ matrix. I will now discuss the structure of this matrix for the cases of two and three generations.

### 2.1.1. Mixing Matrix for Two Generations

In this case, $V$ is a $2 \times 2$ unitary matrix and can be parametrized by one angle and three phases:

$$\begin{aligned} V &= \begin{pmatrix} \cos\theta_{\rm C}\, e^{i\alpha} & \sin\theta_{\rm C}\, e^{i\beta} \\ -\sin\theta_{\rm C}\, e^{i\gamma} & \cos\theta_{\rm C}\, e^{i(\beta+\gamma-\alpha)} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta_{\rm C} & \sin\theta_{\rm C} \\ -\sin\theta_{\rm C} & \cos\theta_{\rm C} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\beta-\alpha)} \end{pmatrix} . \end{aligned} \tag{56}$$

The three phases are not observable, however, as they can be absorbed into a redefinition of the phases of the quark fields $u_{\rm L}$, $c_{\rm L}$, and $s_{\rm L}$ relative to $d_{\rm L}$. After this redefinition, the matrix takes the standard form due to Cabibbo[70]:

$$V_{\rm C} = \begin{pmatrix} \cos\theta_{\rm C} & \sin\theta_{\rm C} \\ -\sin\theta_{\rm C} & \cos\theta_{\rm C} \end{pmatrix} . \tag{57}$$

The Cabibbo angle $\theta_{\rm C}$ can be extracted from $K \to \pi\, e^+ \nu_e$ decay. Experimentally, one finds[52] $\sin\theta_{\rm C} \simeq 0.22$.

### 2.1.2. Mixing Matrix for Three Generations

A $3 \times 3$ unitary matrix can be parametrized by three Euler angles and six phases, five of which can be removed by adjusting the relative phases of the left-handed quark fields. Hence, three angles $\theta_{ij}$ and one observable phase $\delta$ remain in the quark mixing matrix, as was first pointed out by Kobayashi and Maskawa[71]. For completeness, I note that in the general case of $n$ generations, it is easy to show that there are $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}(n-1)(n-2)$ observable phases[72]. Whereas therefore the original Cabibbo matrix was real and had only one parameter, the Kobayashi–Maskawa matrix of the standard model is complex and can be parametrized by four parameters.

The imaginary part of the mixing matrix is necessary to describe CP violation in the standard model. In general, CP is violated in flavour-changing decays if there is no degeneracy of any two quark masses, and if the quantity $J_{\rm CP} \neq 0$, where

$$J_{\rm CP} = |\, {\rm Im}(V_{ij} V_{kl} V_{il}^* V_{kj}^*)\, | \,; \quad i \neq k\,,\ j \neq l \tag{58}$$

is invariant under phase redefinitions of the quark fields. One can show[73] that all CP-violating amplitudes in the standard model are proportional to $J_{\rm CP}$.

I will now discuss two of the most convenient parametrizations of the mixing matrix. The "standard parametrization" recommended by the Particle Data Group[52] is[74]

$$V_{\rm KM} = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{i\delta} & c_{23}\, c_{13} \end{pmatrix} . \tag{59}$$

Here, one uses the short-hand notation $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. Some advantages of this parametrization are the following:

- $|\, V_{ub}| = s_{13}$ is given by a single angle, which experimentally turns out to be very small.
- Because of this, several other entries are given by single angles to an accuracy of better than four digits. They are: $V_{ud} \simeq c_{12}$, $V_{us} \simeq s_{12}$, $V_{cb} \simeq s_{23}$, and $V_{tb} \simeq c_{23}$.
- The CP-violating phase $\delta$ appears together with the small parameter $s_{13}$, making explicit the fact that CP violation in the standard model is a small effect. Indeed, one finds

$$J_{\rm CP} = |\, s_{13}\, s_{23}\, s_{12}\, s_\delta\, c_{13}^2\, c_{23}\, c_{12}\,| \,. \tag{60}$$

For many purposes and applications, it is more convenient to use an approximate parametrization of the Kobayashi–Maskawa matrix, which makes explicit the strong hierarchy that is observed experimentally. Setting $c_{13} = 1$ (experimentally, it is known that $c_{13} > 0.99998$) and neglecting $s_{13}$ compared to terms of order unity, one finds

$$V_{\rm KM} \simeq \begin{pmatrix} c_{12} & s_{12} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} & c_{12}\, c_{23} & s_{23} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -\, c_{12}\, s_{23} & c_{23} \end{pmatrix} . \tag{61}$$

Now denote $s_{12} = \lambda \simeq 0.22$. Experiments indicate that $s_{23} = O(\lambda^2)$ and $s_{13} = O(\lambda^3)$. Hence, it is natural to define $s_{23} = A\,\lambda^2$ and $s_{13}\, e^{-i\delta} = A\,\lambda^3(\rho - i\eta)$, with $A$, $\rho$ and $\eta$ of order unity. An expansion in powers of $\lambda$ then leads to the Wolfenstein parametrization[75]

$$V_{\rm KM} \simeq \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\,\lambda^3(\rho - i\eta) \\ -\,\lambda & 1 - \frac{\lambda^2}{2} & A\,\lambda^2 \\ A\,\lambda^3(1 - \rho - i\eta) & -\,A\,\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \,. \tag{62}$$

It nicely exhibits the hierarchy of the mixing matrix: The entries in the diagonal are close to unity, $V_{us}$ and $V_{cd}$ are of order 20%, $V_{cb}$ and $V_{ts}$ are of order 4%, and $V_{ub}$ and $V_{td}$ are of order 1% and thus the smallest entries in the matrix. Some care has to be taken when one wants to calculate the quantity $J_{\rm CP}$ in the Wolfenstein

parametrization, since the result is of order $\lambda^6$ and thus beyond the accuracy of the approximation. However, taking $i = u$, $j = d$, $k = t$, and $l = b$ in (58), one obtains the correct answer

$$J_{\rm CP} \simeq A^2\,\eta\,\lambda^6 \simeq 1.1 \times 10^{-4} A^2\,\eta\,, \tag{63}$$

which shows that $J_{\rm CP}$ is generically of order $10^{-4}$ for $\lambda \simeq 0.22$. As a consequence, CP violation in the standard model is a small effect.

In principle, the elements in the first two rows of the mixing matrix are accessible in so-called direct (tree-level) processes, i.e. in weak decays of hadrons containing the corresponding quarks. In practice, as I will discuss below, the entries $|\,V_{ud}|$ and $|\,V_{us}|$ are known to an accuracy of better than 1% from such decays, whereas $|\,V_{cd}|$, $|\,V_{cs}|$, and $|\,V_{cb}|$ are known to about 10–20%. The element $|\,V_{ub}|$ has an uncertainty of about a factor of 2. The same is true for $|\,V_{td}|$, which is obtained from a rare process, namely $B_d$–$\bar{B}_d$ mixing. There is no direct information on $|\,V_{ts}|$ and $|\,V_{tb}|$. The large uncertainty in $|\,V_{ub}|$ and $|\,V_{td}|$ translates into an uncertainty of the Wolfenstein parameters $\rho$ and $\eta$. A more precise determination of these parameters will be the challenge to experiments and theory over the next decade.

### *2.2. Experimental Information on $V_{\rm KM}$ from Tree-Level Processes*

In this section, I will briefly review what is known about entries of the Kobayashi–Maskawa matrix from tree-level processes. Rare decays, which are induced by loop-diagrams involving heavy particles, will be discussed later.

#### 2.2.1. Determination of $|\,V_{ud}|$

From a comparison of the very accurate measurements of super-allowed nuclear $\beta$-decay with $\mu$-decay, including a detailed analysis of radiative corrections[76,77], one obtains[52]

$$|\,V_{ud}| = 0.9744 \pm 0.0010\,. \tag{64}$$

#### 2.2.2. Determination of $|\,V_{us}|$

An analysis of $K_{e3}$ decays, i.e. $K^+ \to \pi^0 e^+ \nu_e$ and $K_L^0 \to \pi^- e^+ \nu_e$, including isospin- and SU(3)-breaking effects[78], yields $|\,V_{us}| = 0.2196 \pm 0.0023$. Alternatively, from an analysis of hyperon decays, including SU(3)-breaking corrections[79], one obtains $|\,V_{us}| = 0.222 \pm 0.003$. This leads to the combined value[52]

$$|\,V_{us}| = 0.2205 \pm 0.0018\,. \tag{65}$$

Let me add a note on $K_{e3}$ decays in this context. The fact that the theoretical uncertainty in the description of these exclusive hadronic decays is only about 1% is at first sight surprising. The reasons are the following: The approximate SU(3) flavour symmetry of QCD implies that, at zero momentum transfer, the hadronic

form factor $f_+^{K\to\pi}$ that parametrizes $K \to \pi$ transitions induced by a vector current equals unity, up to symmetry-breaking corrections. The Ademollo–Gatto theorem[80] states that these corrections are of second order in the symmetry-breaking parameter $(m_s - m_q)$, where $q = u$ or $d$. Chiral perturbation theory can be employed to calculate the leading symmetry-breaking corrections (chiral logarithms) in a model-independent way[78]. In my third lecture, I will discuss that there are analogues to all these ingredients in the case of semileptonic $B$-decays.

### 2.2.3. Determination of $|V_{cd}|$

From the combination of data on single charm production in deep-inelastic neutrino–nucleon scattering with the semileptonic branching ratios of charmed mesons, one deduces[52]

$$|V_{cd}| = 0.204 \pm 0.017\,. \tag{66}$$

### 2.2.4. Determination of $|V_{cs}|$

Here the usefulness of deep-inelastic scattering data is limited, since the extraction of $|V_{cs}|$ would depend upon an assumption about the strange quark content of the nucleon. It is better to proceed in analogy to the determination of $|V_{us}|$, i.e. to use the semileptonic $D_{e3}$ decays $D \to \bar{K}\, e^+\nu_e$. The experimental data are compared to various model calculations of the decay width[81–83]. This procedure leads, however, to a significant amount of model dependence. The result is[52]

$$|V_{cs}| = 1.00 \pm 0.20\,. \tag{67}$$

You may wonder why the theoretical description of the decay width is so uncertain. The reason is that, in general, our capabilities to calculate hadronic form factors in QCD are rather limited. An understanding of the nonperturbative confining forces would be necessary to make the connection between quark and hadron properties, which is a prerequisite for any calculation of hadronic matrix elements. In the case of $K_{e3}$ decays, a symmetry of QCD helps us eliminate most of the hadronic uncertainties. For $D_{e3}$ decays, there is no such flavour symmetry. Hence, one has to rely on phenomenological models. Let me briefly mention some of them:

- Isgur, Scora, Grinstein, and Wise (ISGW) have proposed a nonrelativistic constituent quark model[83]. They solve the Schrödinger equation for a Coulomb plus linear potential to obtain meson wave functions. Weak decay form factors are calculated at maximum momentum transfer, where both mesons have a common rest frame, by computing overlap integrals of the wave functions of the initial and final meson. The nonrelativistic approximation breaks down if $q^2 \ll q^2_{\rm max}$.
- Bauer, Stech, and Wirbel (BSW) have constructed a relativistic model[81], which uses light-cone wave functions to calculate weak decay form factors

at zero momentum transfer. In order to extrapolate to $q^2 > 0$, they assume nearest-pole dominance.

- Körner and Schuler[84] (KS) use a variation of the BSW approach, in which the $q^2$ dependence of the form factors is adjusted according to asymptotic QCD power-counting rules[85].

- There are a variety of other models, for instance by Suzuki[86], Altomari and Wolfenstein[87], and Grinstein et al.[82], which are more or less closely related to one of the above.

Disadvantages of these models are that their relation to the underlying theory of QCD is not obvious, that it is hard to obtain an estimate of their intrinsic uncertainty or range of applicability, that they rely on ad hoc assumptions (for instance, concerning the $q^2$ dependence of form factors), and that it is difficult to improve them in a systematic way.

Another stream of research focuses on analytical or numerical approaches that bear a closer relation to field theory. QCD sum rules[29] provide a fully relativistic approach based on QCD. They rely, however, on the assumption of quark–hadron duality and need some phenomenological input. Nonperturbative effects are modelled in a simple way by introducing few universal numbers, the so-called vacuum condensates. There have been extensive applications of sum rules to the study of meson weak decay form factors[88–96]. These analyses can be done for all physical values of $q^2$. However, in spite of the many successes of QCD sum rules it must be said that the only known approach to low-energy QCD that truly starts from first principles is lattice gauge theory. For more details on this, I refer to the lectures by P. Lepage in this volume. Let me just mention that, as far as weak decay form factors are concerned, these computations are extremely complex and CPU-consuming. They are not yet competitive with (less rigorous) analytical approaches.

### 2.2.5. Determination of $|V_{ub}|$

Since $b \to u$ transitions are strongly suppressed in nature, their discovery in inclusive $B \to X_u \ell \bar{\nu}$ decays was one of the breakthroughs in $B$-physics in recent years[3–6]. In order to account for CP violation in the standard model, it is necessary that $V_{ub}$ be different from zero; otherwise $J_{\rm CP} = 0$, and the observed CP violation in the kaon system[97] could not be explained by the Kobayashi–Maskawa mechanism.

The first direct observation of a $b \to u$ transition was provided by two fully reconstructed events, one $B^0 \to \pi^+ \mu^- \bar{\nu}$ decay and one $B^+ \to \omega \mu^+ \nu$ decay, reported by the ARGUS collaboration[5]. However, the number of exclusive charmless $B$-decays that have been observed so far is too small to obtain a reasonable determination of $|V_{ub}|$. A high luminosity $B$-factory would improve this situation. Given enough statistics, the idea is to obtain a model-independent measurement of $|V_{ub}|$ by using heavy quark symmetry to relate the decay form factors in $B \to X_u \ell \bar{\nu}$

and $D \to X_d \ell \bar{\nu}$ transitions[98], at the same value of the recoil energy of the light hadron $X_q$. In the limit of infinite heavy quark masses ($m_b, m_c \to \infty$), the ratio of these form factors is fixed in a model-independent way. In the real world, however, there are nonperturbative power corrections to this limit. As a consequence, even in an ideal measurement the ratio $|V_{ub}/V_{cd}|$ can only be determined up to hadronic corrections of order

$$1 + c\left(\frac{\Lambda_{\rm QCD}}{m_c} - \frac{\Lambda_{\rm QCD}}{m_b}\right) + \dots , \tag{68}$$

with a coefficient $c$ of order unity. Model calculations[99,100] show that these corrections are of order 15% for $X = \pi$ or $\rho$. I believe that it is fair to say that even at a high-luminosity $B$-factory, the prospects for getting a determination of $|V_{ub}|$ from exclusive decays, which is more reliable than that from inclusive decays, are rather limited. Nevertheless, it is certainly worth while to pursue this strategy as an alternative.

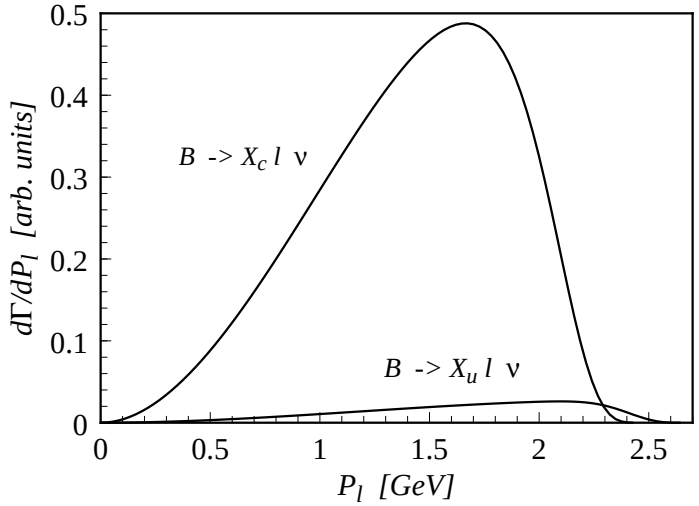


Figure 6: Sketch of the lepton spectrum in inclusive semileptonic $B$-decays. I assume $|V_{ub}/V_{cb}| \simeq 0.08$. The $b \to u$ signal has been multiplied by a factor of 5 to be visible on this plot.

The present determinations of $|V_{ub}|$ are based on measurements of the lepton momentum spectrum in inclusive $B \to X_q \ell \bar{\nu}$ decays, where $X_q$ is any hadronic state containing a $q$-quark, with $q = c$ or $u$. The expected signal is shown in Fig. 6. Over most of the kinematic region, the spectrum is dominated by $b \to c$ transitions, which are strongly enhanced with respect to $b \to u$ decays. The only place to observe charmless transitions is in a small window above the kinematic limit for $B \to D \ell \bar{\nu}$, but below the kinematic limit for $B \to \pi \ell \bar{\nu}$ decays:

$$\frac{m_B}{2}\left(1 - \frac{m_D^2}{m_B^2}\right) \le p_\ell \le \frac{m_B}{2}\left(1 - \frac{m_\pi^2}{m_B^2}\right), \tag{69}$$

i.e. 2.31 GeV $\leq p_\ell \leq$ 2.64 GeV. Indeed, the ARGUS and CLEO collaborations have reported signals in this region[3-6]. An extraction of $|V_{ub}|$ from these data is difficult, however, as the shape of the spectrum close to the kinematic endpoint is dominated by nonperturbative effects. This can be seen as follows. The conventional description of inclusive decays of hadrons containing a heavy quark starts from the free-quark decay model, in which the hadronic decay $B \to X_q \ell \bar{\nu}$ is modelled by the quark decay $b \to q \ell \bar{\nu}$. One usually calculates the decay distributions to order $\alpha_s$ in perturbation theory by including the effects of real and virtual gluon emission[101-103]. Computing then the lepton spectrum, one finds that the kinematic limit for $b \to u$ decays is given by $p_\ell^{\rm max} = m_b/2 \simeq 2.35$ GeV, where I neglect the $u$-quark mass and take $m_b \simeq 4.7$ GeV for the sake of argument. The point is that the region in which the $b \to u$ signal is observed experimentally is almost not populated in the free-quark decay model. Nonperturbative bound-state effects, such as the motion of the $b$-quark inside the $B$-meson, are responsible for the population of the spectrum beyond the parton model endpoint. The way in which such effects are incorporated into phenomenological approaches[104,105] is to a large extent model-dependent. Only very recently has there been progress towards a model-independent description of the distributions near the kinematic endpoint in the context of QCD[106-110]. It has been shown that the leading nonperturbative effects can be parametrized in terms of a universal structure function, which describes the momentum distribution of the $b$-quark inside the $B$-meson. The hope is that it will be possible to get an accurate prediction for this function using various theoretical approaches. I believe that, within a year or two, it should be possible to reduce the theoretical uncertainty in the analysis of the inclusive decay spectrum by a factor of 2 or so.

Alternatively, one can describe the lepton spectrum close to the endpoint as the sum of contributions from semileptonic $B$-decays into exclusive final states. These exclusive decays are then described using one of the various bound-state models[81-84] discussed above. Of course, there is again a strong model dependence associated with this approach.

Because of these theoretical uncertainties, the current value of $|V_{ub}|$ has a model dependence of almost a factor of 2. In 1991, the ARGUS collaboration obtained[5] $|V_{ub}/V_{cb}| = 0.11 \pm 0.012$, using the approach of Altarelli et al.[104], which is based on the parton model. Using instead bound-state models to describe the spectrum by a sum over several exclusive decay channels, values for $|V_{ub}/V_{cb}|$ between 0.10 and 0.22 are obtained. However, the most recent data reported by the CLEO collaboration lead to significantly smaller values[6]: $|V_{ub}/V_{cb}| = 0.076 \pm 0.008$ using the approach of Altarelli et al.[104], and $0.05 < |V_{ub}/V_{cb}| < 0.11$ using bound-state models. Thus, I think a reasonable value to quote is

$$\left| \frac{V_{ub}}{V_{cb}} \right| = 0.08 \pm 0.03 \,. \tag{70}$$

2.2.6. Determination of $|V_{cb}|$

As for $|V_{cb}|$, it can be extracted from both inclusive and exclusive $B$-decays. I start with a discussion of inclusive decays. The idea is to compare the total semileptonic branching ratio to the parton model formula

$$\mathrm{Br}(B \to X_q \ell \bar{\nu}) = \frac{G_F^2 m_b^5}{192\pi^3} \tau_B \left\{ \eta_c |V_{cb}|^2 f\left(\frac{m_c^2}{m_b^2}\right) + \eta_u |V_{ub}|^2 \right\}, \tag{71}$$

where $\eta_c$ and $\eta_u$ contain the short-distance QCD corrections, and

$$f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \tag{72}$$

is a phase-space correction due to the mass of the charm quark. The QCD correction factor for $b \to c$ decays is[101,102] $\eta_c \simeq 0.87$. The contribution from $b \to u$ transitions is very small and, at the present level of accuracy, can be neglected.

Let me note that it has recently become clear how to compute in a systematic way the nonperturbative bound-state corrections to the parton model result (71). For the total inclusive decay rates, these corrections turn out to be of order[111–119] $(\Lambda_{\rm QCD}/m_b)^2$. Numerically, they are very small and play only a minor role in the analysis[120].

The disadvantage of using inclusive decays to extract $|V_{cb}|$ is that there are substantial theoretical uncertainties due to the fact that the total rate scales as the fifth power of the bottom quark mass. For instance, an uncertainty of only $\pm 150$ MeV in the value of $m_b$ translates into an uncertainty of $\pm 8\%$ in the value of $|V_{cb}|$. For this reason, the last time the particle data group[121] used inclusive decays to obtain a value for $|V_{cb}|$ was in 1988. A compilation of more recent analyses of inclusive decay spectra has been given by Stone[122]. He finds

$$|V_{cb}| \left(\frac{\tau_B}{1.5 \text{ ps}}\right)^{1/2} = 0.040 \pm 0.005\,, \tag{73}$$

where I have normalized the result to a value of the $B$-meson lifetime that is close to the world average $\tau_{B^0} = (1.48 \pm 0.10)$ ps reported last year by Danilov[123], as well as to the most recent value $\tau_{B^0} = (1.52 \pm 0.13)$ ps obtained from an average of LEP results[124]. The error quoted in (73) does not take into account the theoretical uncertainty in the value of $m_b$.

Ultimately, a more precise determination must come from the analysis of exclusive decay modes. With the development of the heavy quark expansion, it has become clear that certain symmetries of QCD in the heavy quark limit can help to remove much of the model dependence from the theoretical description of exclusive semileptonic $B$-decays. This will be explained in detail in my third lecture. In particular, for the decay mode $B \to D^* \ell \bar{\nu}$ the theoretical uncertainties in the momentum distribution of the $D^*$-meson close to the zero recoil limit can be controlled

at the level of a few per cent[125]. The most recent analysis of this spectrum by the CLEO collaboration leads to the value[126]

$$|V_{cb}| \left(\frac{\tau_B}{1.5 \text{ ps}}\right)^{1/2} = 0.039 \pm 0.006\,, \tag{74}$$

which I will use in the analysis below. In contrast to (73), here the uncertainty is dominated by the experimental systematic errors and, to a lesser extent, the statistical ones. The theoretical uncertainty is well below 10%. There is thus room for a significant improvement of the accuracy within the next few years, as more data become available.

Finally, let me mention that by combining the above result with the value of $|V_{us}|$ given in (65), one obtains

$$A = \left|\frac{V_{cb}}{V_{us}^2}\right| = 0.80 \pm 0.12 \tag{75}$$

for the Wolfenstein parameter $A$.

### *2.3. Unitarity Constraints*

The fact that the quark mixing matrix must be unitary can be employed to put limits on the elements that are not easily accessible to experiments. One has to assume, however, that there are only three quark generations. Unitarity then implies that

$$|V_{tj}|^2 = 1 - |V_{uj}|^2 - |V_{cj}|^2\,. \tag{76}$$

Inserting here the known values of $|V_{uj}|$ and $|V_{cj}|$, one obtains

$$\begin{aligned} 0.003 &< |V_{td}| < 0.016\,, \\ 0.032 &< |V_{ts}| < 0.048\,, \\ 0.9991 &< |V_{tb}| < 0.9994\,. \end{aligned} \tag{77}$$

Note, in particular, that $|V_{tb}|$ is constrained to an accuracy of better than $10^{-3}$.

Unitarity also imposes a rather tight constraint on $|V_{cs}|$. Using that

$$|V_{cs}|^2 = 1 - |V_{cd}|^2 - |V_{cb}|^2\,, \tag{78}$$

one finds

$$0.976 < |V_{cs}| < 0.981\,. \tag{79}$$

This should be compared to the value (67) extracted from a direct measurement.

### *2.4. The Unitarity Triangle*

A simple but beautiful way to visualize the implications of unitarity is provided by the so-called unitarity triangle[127], which uses the fact that the unitarity equation

$$V_{ij}\, V_{ik}^* = 0 \qquad (j \neq k) \tag{80}$$

can be represented as the equation of a closed triangle in the complex plane. There are six such triangles, all of which have the same area[128]

$$|A_\Delta| = \frac{1}{2}\, J_{\rm CP}\,. \tag{81}$$

Under phase reparametrizations of the quark fields, the triangles change their orientation in the complex plane, but their shape remains unaffected.

Most useful from the phenomenological point of view is the triangle relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\,, \tag{82}$$

since it contains the most poorly known entries in the Kobayashi–Maskawa matrix. It has been widely discussed in the literature[127–134]. In the standard parametrization, $V_{cd}V_{cb}^*$ is real, and the unitarity triangle has the form shown in Fig. 7a. It is useful to rescale the triangle by dividing all sides by $V_{cd}V_{cb}^*$. The rescaled triangle is shown in Fig. 7b. It has the coordinates $(0,0)$, $(1,0)$, and $(\rho,\eta)$, where $\rho$ and $\eta$ appear in the Wolfenstein parametrization (62). Unitarity amounts to the statement that the triangle is closed, and CP is violated when the area of the triangle does not vanish, i.e. when all the angles are not zero.

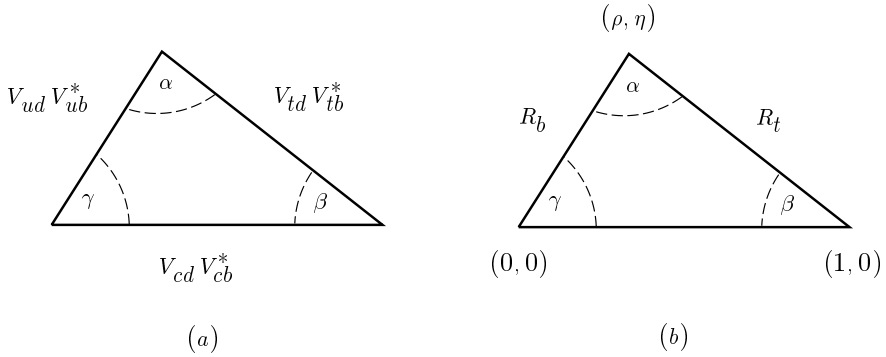


Figure 7: (a) The unitarity triangle; (b) its rescaled form in the $\rho$–$\eta$ plane. The angle $\gamma$ coincides with the phase $\delta$ of the standard parametrization (59).

To determine the shape of the triangle, one can aim for measurements of the two sides $R_b$ and $R_t$, and of the angles $\alpha$, $\beta$, and $\gamma$. So far, experimental information is available only on the sides of the triangle. The current value of $|\,V_{ub}|$ in (70) implies

$$0.24 < R_b < 0.53\,. \tag{83}$$

As mentioned above, the uncertainty in this number is mainly a theoretical one, so any significant improvement must originate from new developments in the theory of inclusive $B$-decays.

Let me now discuss the determination of $|V_{td}|$ and of the side $R_t$ of the unitarity triangle. The process of interest here is the mixing between $B_d$- and $\bar{B}_d$-mesons, which in the standard model is a rare process mediated through box diagrams with a virtual top quark exchange. The ARGUS and CLEO collaborations have reported measurements[135,136] of the mixing parameter $x_d$ defined as $x_d = \Delta m_{B_d}\,\tau_{B_d}$, where $\Delta m_B$ denotes the mass difference between the mass eigenstates in the $B_d$–$\bar{B}_d$ system. The weighted average of their results is $x_d = 0.68 \pm 0.08$. Combining this with the $B_d$-meson lifetime[123], $\tau_{B_d} = (1.48 \pm 0.10)$ ps, one obtains $\Delta m_B = (0.46 \pm 0.06)$ ps$^{-1}$. Recently, direct measurements of $\Delta m_B$ have been reported by some of the LEP experiments. The average value presented at the Moriond Conference is[137] $\Delta m_B = (0.519^{+0.063}_{-0.059})$ ps$^{-1}$. I combine these results to obtain

$$\Delta m_B = (0.49 \pm 0.04)\ \text{ps}^{-1}\,. \tag{84}$$

The theoretical expression for $\Delta m_B$ in the standard model is

$$\Delta m_B = \frac{G_F^2 m_W^2}{6\pi^2}\,\eta_{\rm QCD}\,m_B\,(B_B f_B^2)\,S(m_t/m_W)\,|V_{td}V_{tb}^*|^2\,, \tag{85}$$

where $\eta_{\rm QCD} \simeq 0.55$ contains the next-to-leading order QCD corrections[138], and $S(m_t/m_W)$ is a function of the top quark mass[139,140]. For 100 GeV $< m_t <$ 300 GeV, it can be approximated by[141] $S(m_t/m_W) \simeq 0.784\,(m_t/m_W)^{1.52}$. The product $(B_B f_B^2)$ parametrizes the hadronic matrix element of a local four-quark operator between $B$-meson states. Recently, there has been a lot of activity and improvement in theoretical calculations of the $B$-meson decay constant $f_B$, using both lattice gauge theory and QCD sum rule calculations. For a summary and discussion of the extensive literature on this subject, I refer to the review articles by Buras and Harlander[68] and by myself[25]. Here I shall use the range

$$B_B^{1/2} f_B = (200 \pm 40)\ \text{MeV}\,, \tag{86}$$

which covers most theoretical predictions. Solving (85) for $|V_{td}|$, one obtains

$$|V_{td}| = 0.0088 \left(\frac{200\ \text{MeV}}{B_B^{1/2} f_B}\right) \left(\frac{170\ \text{GeV}}{m_t}\right)^{0.76} \left(\frac{\Delta m_B}{0.5\ \text{ps}^{-1}}\right)^{1/2}\,. \tag{87}$$

Using the numbers given above, as well as $m_t = (170 \pm 30)$ GeV, I find

$$|V_{td}| = 0.0087 \pm 0.0023\,. \tag{88}$$

The corresponding range of values for $R_t$ is

$$0.75 < R_t < 1.44\,. \tag{89}$$

The error in this value will hopefully be reduced soon, with a measurement of the top quark mass at Fermilab. Further improvements could come from a measurement

of $f_B$ in $B^+ \to \tau^+\nu_\tau$ decays, or from the observation of $B_s$–$\bar{B}_s$ mixing. In the latter case, one could use the relation

$$\frac{x_s}{x_d} = \left|\frac{V_{ts}}{V_{td}}\right|^2 \times \Big\{1 + \text{SU(3)-breaking corrections}\Big\} \tag{90}$$

to obtain a rather clean measurement of $|V_{td}|$. However, in the standard model one expects large values $x_s \sim 10$–$30$, which will be very hard to observe in near-future experiments.

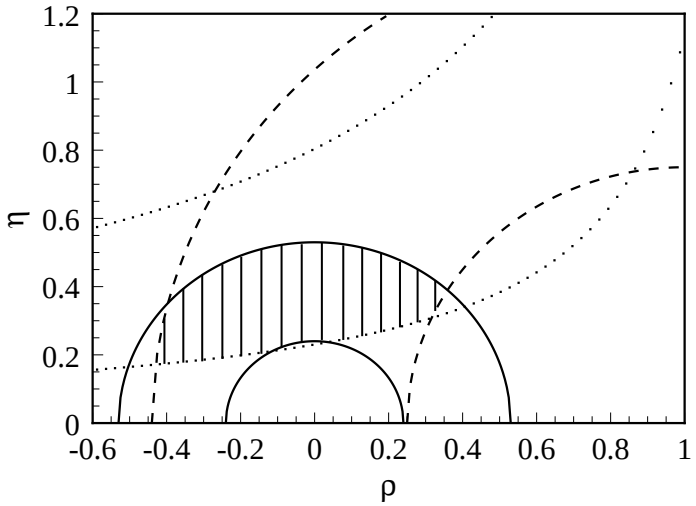


Figure 8: Experimental constraints on the unitarity triangle in the $\rho$–$\eta$ plane. The region between the solid (dashed) circles is allowed by the measurement of $R_b$ ($R_t$) discussed above. The dotted curves show the constraint following from the measurement of the $\varepsilon$ parameter in the kaon system. The shaded region shows the allowed range for the tip of the unitarity triangle. The base of the triangle has the coordinates $(0,0)$ and $(1,0)$.

In Fig. 8, I show the constraints imposed by the above results for $R_b$ and $R_t$ in the $\rho$–$\eta$ plane. Another constraint can be obtained from the study of CP violation in $K$-decays.

### *2.5. CP Violation in K-Decays*

The experimental result on the parameter $\varepsilon$ that measures CP violation in $K$–$\bar{K}$ mixing implies that the unitarity triangle lies in the upper half plane, provided the so-called $B_K$ parameter is positive. Most theoretical predictions[142–150] fall in the range $B_K = 0.65 \pm 0.20$, supporting this assertion. The arising constraint in the $\rho$–$\eta$

plane has the form of a hyperbola approximately given by[68]

$$\eta \left[ (1-\rho)\, A^2 \left( \frac{m_t}{m_W} \right)^{1.52} + P_c \right] A^2\, B_K = 0.50\,, \tag{91}$$

where $P_c \simeq 0.66$ corresponds to the contributions from box diagrams containing charm quarks, and $A = 0.80 \pm 0.12$ according to (75). I have included this bound in Fig. 8.

In principle, the measurement of the ratio $\mathrm{Re}(\varepsilon'/\varepsilon)$ in the kaon system[151,152] could provide a determination of $\eta$ independent of $\rho$. In practice, however, the theoretical calculations[153–155] of this ratio are affected by large uncertainties, so that there currently is no useful bound to be derived. Further information on the parameters $\rho$ and $\eta$ could be extracted from rare $K$-decays, such as $K_L \to \pi^0 \nu\, \bar{\nu}$, $K^+ \to \pi^+ \nu\, \bar{\nu}$, and $K_L \to \mu^+ \mu^-$. The corresponding branching ratios vary between $10^{-11}$ and $10^{-10}$. The experimental detection of such decays will be very hard.

### *2.6. Potential Improvements in the Determination of $\rho$ and $\eta$*

The main conclusion to be drawn from Fig. 8 is that, given the present theoretical and experimental uncertainties in the analysis of charmless $B$-decays, $B$–$\bar{B}$ mixing, and CP violation in the kaon system, there is still a large region in parameter space that is allowed for the Wolfenstein parameters $\rho$ and $\eta$. This has important implications. For instance, the allowed region for the angle $\beta$ of the unitarity triangle (see Fig. 7) is $6.9^\circ < \beta < 31.8^\circ$, corresponding to

$$0.24 < \sin 2\beta < 0.90\,. \tag{92}$$

Below I will discuss that the CP asymmetry in the decay $B \to \psi\, K_S$, which is one of the favoured modes to search for CP violation at a future $B$-factory, is proportional to $\sin 2\beta$. Obviously, the prospects for discovering CP violation with such a machine crucially depend on whether $\sin 2\beta$ is closer to the upper or lower bound in (92). A more reliable determination of the shape of the unitarity triangle is thus of the utmost importance.

Let me briefly discuss what I think are the potential improvements in this determination in the near future. It is not unrealistic to assume the following:

- The top quark will be discovered at Fermilab.
- The uncertainty in the value of $|\, V_{cb}|$ will be reduced to $\pm 0.04$, mainly through a better control of the experimental systematic errors.
- Both theoretical and experimental improvements will reduce the uncertainty in the determination of $|\, V_{ub}|$ by a factor of 2.
- Improved theoretical calculations, in particular using lattice gauge theory, will allow the determination of the nonperturbative parameters $B_B^{1/2} f_B$ and $B_K$ with an accuracy of 10%.

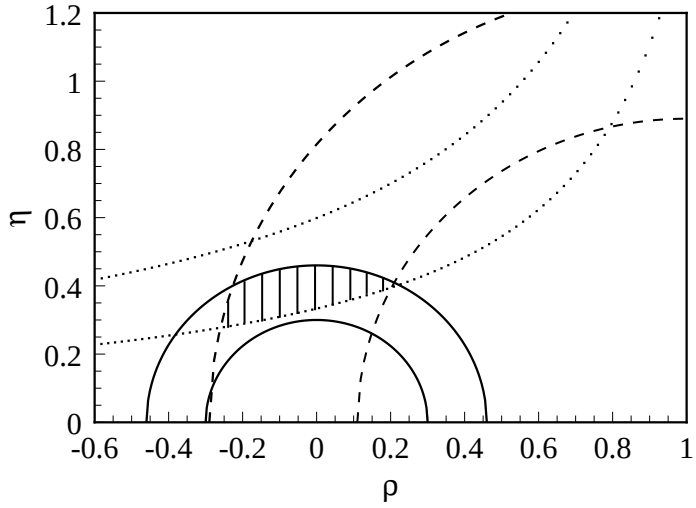


Figure 9: Constraints on the unitarity triangle obtained assuming a more precise determination of the various input parameters, as explained in the text.

For the purpose of illustration, I shall assume that $m_t = 170$ GeV, $|V_{cb}| = 0.039 \pm 0.004$, $|V_{ub}/V_{cb}| = 0.080 \pm 0.015$, $B_B^{1/2} f_B = (200 \pm 20)$ MeV, and $B_K = 0.65 \pm 0.07$. This leads to $R_b = 0.38 \pm 0.08$ and $R_t = 1.09 \pm 0.20$. Similarly, the constraint derived from the measurement of $\varepsilon$ becomes more restrictive. This fictitious scenario is illustrated in Fig. 9. Obviously, the parameters of the triangle (the value of $\eta$, in particular) would be much more constrained than they are at present. Notice also that once a sufficient accuracy is obtained, there is a potential for tests of the standard model. If the three bands in Fig. 9 did not overlap, this would be an indication of new physics.

### *2.7. Rare B-Decays*

Let me briefly touch upon the interesting subject of rare $B$-decays, i.e. decays induced by loop diagrams. There are nice review articles on this subject by Bertolini[156] and Ali[157]. Rare $B$-decays are dominated by short-distance physics, namely the top quark contribution in penguin and box diagrams. The decay rates are usually rather sensitive to the mass of the top quark[141]. Long-distance effects are much less important than in rare $K$-decays. Hence, rare $B$-decays are "clean" from the theoretical point of view. They are ideal to determine the parameters of the unitarity triangle. In addition, these decays can provide important tests of the standard model, since they are often sensitive probes of new physics.

So far, the only rare $B$-decay that has been observed is[7] $B \to K^* \gamma$. In general, decays of the type $B \to X_s \gamma$ are mediated by penguin diagrams such as the ones shown in Fig. 10. These decays receive very large, but calculable, short-distance

QCD corrections[158–164], which strongly enhance the decay rate. In addition, it is important to take into account the effect of gluon bremsstrahlung, which affects the total decay rate as well as the photon spectrum[165]. All such effects are reasonably well under control for the inclusive $B \to X_s\,\gamma$ decay rate. For $m_t = 170 \pm 30$ GeV, the prediction is[68]

$$\mathrm{Br}(B \to X_s\,\gamma) = (2.8\text{–}4.2) \times 10^{-4}\,. \tag{93}$$

Of more interest from the experimental point of view are exclusive rare decay modes. Since $B \to K\,\gamma$ is forbidden by angular momentum conservation, the lightest final state that can appear is $K^*\gamma$. To obtain from (93) a prediction for $\mathrm{Br}(B \to K^*\gamma)$ requires an estimate of the ratio

$$R_{K^*} = \frac{\Gamma(B \to K^*\gamma)}{\Gamma(B \to X_s\,\gamma)}\,. \tag{94}$$

This is where hadronic uncertainties enter the analysis. Estimates of $R_{K^*}$ presented in the literature vary between 4% and 40%. I believe that a reasonable value is[165–168] $R_{K^*} \simeq 10\text{–}15\%$. This leads to

$$\mathrm{Br}(B \to K^*\gamma) \sim (3\text{–}6) \times 10^{-5}\,. \tag{95}$$

This is in agreement with the result reported by the CLEO collaboration[7]:

$$\mathrm{Br}(B \to K^*\gamma) = (4.5 \pm 1.9 \pm 0.9) \times 10^{-5}\,. \tag{96}$$

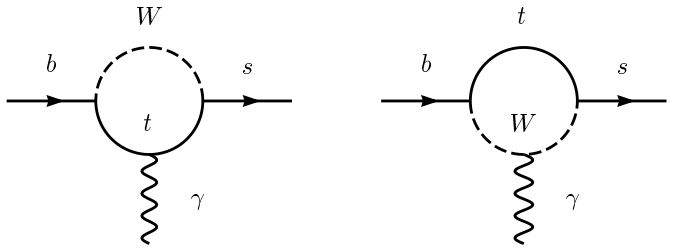


Figure 10: Penguin diagrams contributing to the rare decay $B \to X_s\,\gamma$.

Further rare $B$-decays, for which there are "clean" theoretical predictions (which depend, however, on the top quark mass), include $B \to X_s\,\nu\,\bar\nu$ with a branching ratio of $10^{-5}$–$10^{-4}$, $B \to X_s\,\ell\,\bar\ell$ with a branching ratio of about $10^{-5}$, and $B_s \to \ell\,\bar\ell$ with a branching ratio of about $10^{-7}$.

### *2.8. Measurements of the Angles of the Unitarity Triangle*

Let me finish this short introduction into the phenomenology of rare decays with a very beautiful application, namely the direct measurement of CP violation

in decays of neutral $B$-mesons into CP eigenstates $f$. For these processes, there are often no (or only very small) hadronic uncertainties. The CP asymmetries arise from an interference of mixing and decay. When such a decay is dominated by a single weak decay amplitude, the time-integrated asymmetry obeys a very simple relation to one of the angles $\varphi$ of the unitarity triangle[68,69,129]:

$$\frac{\int_0^\infty \mathrm{d}t \left[\Gamma(B^0(t) \to f) - \Gamma(\bar{B}^0(t) \to f)\right]}{\int_0^\infty \mathrm{d}t \left[\Gamma(B^0(t) \to f) + \Gamma(\bar{B}^0(t) \to f)\right]} = \pm \sin(2\varphi)\, \frac{x}{1+x^2}\,, \tag{97}$$

where $x = \Delta m_B\, \tau_B$. Examples of such decays are:

$$\begin{aligned}
B_d &\to \psi\, K_S && (-\sin 2\beta)\,, \\
B_d &\to \phi\, K_S && (-\sin 2\beta)\,, \\
B_d &\to D^+ D^- && (-\sin 2\beta)\,, \\
B_d &\to \pi^+ \pi^- && (\sin 2\alpha)\,, \\
B_s &\to \rho\, K_S && (-\sin 2\gamma)\,, \\
B_s &\to \phi\, K_S && (\sin 2\beta)\,.
\end{aligned} \tag{98}$$

The decay mode $B_d \to \psi\, K_S$ is particularly "clean" and is often referred to as the "gold-plated" mode of a future $B$-factory.

### *2.9. Exercises*

- Show that the generalization of the Kobayashi–Maskawa matrix to $n$ quark generations can be parametrized by $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}(n-1)(n-2)$ observable phases.
- Show that the quantity $J_{\mathrm{CP}}$ defined in (58) is invariant under phase redefinitions of the quark fields. Verify that in the standard parametrization $J_{\mathrm{CP}}$ is given by (60).
- Derive that the maximum lepton momentum in the decay $B \to X\, \ell\, \bar{\nu}$ is given by $\frac{1}{2}m_B(1 - m_X^2/m_B^2)$ [cf. (69)].
- Convince yourself that all six unitarity triangles have the same area (81).
- Show that the decay $B \to K\, \gamma$ is forbidden by angular momentum conservation.

## 3. Heavy Quark Symmetry

In this lecture, I give an introduction to one of the most active and exciting recent developments in theoretical particle physics: the discovery of heavy quark symmetry and the development of the heavy quark effective field theory. For a more detailed description, I refer to my recent review article in *Physics Reports*[25]. Earlier introductions into this field were given by Georgi[169], Grinstein[170], Isgur and Wise[171], and Mannel[172].

### *3.1. The Physical Picture*[8–13]

There are several reasons why the strong interactions of systems containing heavy quarks are easier to understand than those of systems containing only light quarks. The first is asymptotic freedom, the fact that the effective coupling constant of QCD becomes weak in processes with large momentum transfer, corresponding to interactions at short-distance scales[38,39]. At large distances, on the other hand, the coupling becomes strong, leading to nonperturbative phenomena such as the confinement of quarks and gluons on a length scale $R_{\rm had} \sim 1/\Lambda_{\rm QCD} \sim 1$ fm, which determines the typical size of hadrons. Roughly speaking, $\Lambda_{\rm QCD} \sim 0.2$ GeV is the energy scale that separates the regions of large and small coupling constant. When the mass of a quark $Q$ is much larger than this scale, $m_Q \gg \Lambda_{\rm QCD}$, $Q$ is called a heavy quark. The quarks of the standard model fall naturally into two classes: up, down and strange are light quarks, whereas charm, bottom and top are heavy quarks[‡]. For heavy quarks, the effective coupling constant $\alpha_s(m_Q)$ is small, implying that on length scales comparable to the Compton wavelength $\lambda_Q \sim 1/m_Q$ the strong interactions are perturbative and much like the electromagnetic interactions. In fact, the quarkonium systems $(\bar{Q}Q)$, whose size is of order $\lambda_Q/\alpha_s(m_Q) \ll R_{\rm had}$, are very much hydrogen-like. Since the discovery of asymptotic freedom, their properties could be predicted[173] before the observation of charmonium, and later of bottomonium states.

Things are more complicated for systems composed of a heavy quark and other light constituents. The size of such systems is determined by $R_{\rm had}$, and the typical momenta exchanged between the heavy and light constituents are of order $\Lambda_{\rm QCD}$. The heavy quark is surrounded by a most complicated, strongly interacting cloud of light quarks, antiquarks, and gluons. This cloud is sometimes referred to as the "brown muck", a term invented by Isgur to emphasize that the properties of such systems cannot be calculated from first principles (at least not in a perturbative way). In this case, it is the fact that $m_Q \gg \Lambda_{\rm QCD}$, or better $\lambda_Q \ll R_{\rm had}$, i.e. the fact that the Compton wavelength of the heavy quark is much smaller than the size of the hadron, which leads to simplifications. To resolve the quantum numbers of the heavy quark would require a hard probe. The soft gluons which couple to the

[‡]Ironically, the top quark is of no relevance to my discussion here, since it is too heavy to form hadronic bound states before it decays.

"brown muck" can only resolve distances much larger than $\lambda_Q$. Therefore, the light degrees of freedom are blind to the flavour (mass) and spin orientation of the heavy quark. They only experience its colour field, which extends over large distances because of confinement. In the rest frame of the heavy quark, it is in fact only the electric colour field that is important; relativistic effects such as colour magnetism vanish as $m_Q \to \infty$. Since the heavy quark spin participates in interactions only through such relativistic effects, it decouples. That the heavy quark mass becomes irrelevant can be seen as follows: As $m_Q \to \infty$, the heavy quark and the hadron that contains it have the same velocity. In the rest frame of the hadron, the heavy quark is at rest, too. The wave function of the "brown muck" follows from a solution of the field equations of QCD subject to the boundary condition of a static triplet source of colour at the location of the heavy quark. This boundary condition is independent of $m_Q$, and so is the solution for the configuration of the light degrees of freedom.

It follows that, in the $m_Q \to \infty$ limit, hadronic systems which differ only in the flavour or spin quantum numbers of the heavy quark have the same configuration of their light degrees of freedom. Although this observation still does not allow us to calculate what this configuration is, it provides relations between the properties of such particles as the heavy mesons $B$, $D$, $B^*$ and $D^*$, or the heavy baryons $\Lambda_b$ and $\Lambda_c$ (to the extent that corrections to the infinite quark mass limit are small in these systems). Isgur and Wise realized that these relations result from new symmetries of the effective strong interactions of heavy quarks at low energies[13]. The configuration of light degrees of freedom in a hadron containing a single heavy quark $Q(v, s)$ with velocity $v$ and spin $s$ does not change if this quark is replaced by another heavy quark $Q'(v, s')$ with different flavour or spin, but with the same velocity. Both heavy quarks lead to the same static colour field. It is not necessary that the heavy quarks $Q$ and $Q'$ have similar masses. What is important is that their masses are large compared to $\Lambda_{\rm QCD}$. For $N_h$ heavy quark flavours, there is thus an SU($2N_h$) spin-flavour symmetry group. These symmetries are in close correspondence to familiar properties of atoms: The flavour symmetry is analogous to the fact that different isotopes have the same chemistry, since to good approximation the wave function of the electrons is independent of the mass of the nucleus. The electrons only see the total nuclear charge. The spin symmetry is analogous to the fact that the hyperfine levels in atoms are nearly degenerate. The nuclear spin decouples in the limit $m_e/m_N \to 0$.

The heavy quark symmetry is an approximate symmetry, and corrections arise since the quark masses are not infinite. These corrections are of order $\Lambda_{\rm QCD}/m_Q$. The condition $m_Q \gg \Lambda_{\rm QCD}$ is necessary and sufficient for a system containing a heavy quark to be close to the symmetry limit. In many respects, heavy quark symmetry is complementary to chiral symmetry, which arises in the opposite limit of small quark masses, $m_q \ll \Lambda_{\rm QCD}$. There is an important distinction, however. Whereas chiral symmetry is a symmetry of the QCD Lagrangian in the limit of vanishing quark masses, heavy quark symmetry is not a symmetry of the Lagrangian (not even an approximate one), but rather a symmetry of an effective theory which

is a good approximation to QCD in a certain kinematic region. It is realized only in systems in which a heavy quark interacts predominantly by the exchange of soft gluons. In such systems the heavy quark is almost on-shell; its momentum fluctuates around the mass shell by an amount of order $\Lambda_{\rm QCD}$. The corresponding changes in the velocity of the heavy quark vanish as $\Lambda_{\rm QCD}/m_Q \to 0$. The velocity becomes a conserved quantity and is no longer a dynamical degree of freedom[21].

Results derived based on heavy quark symmetry are model-independent consequences of QCD in a well-defined limit. The symmetry-breaking corrections can, at least in principle, be studied in a systematic way. To this end, it is however necessary to recast the QCD Lagrangian for a heavy quark,

$$\mathcal{L}_Q = \bar{Q}\,(i\not{D} - m_Q)\,Q\,, \tag{99}$$

into a form suitable for taking the limit $m_Q \to \infty$.

### *3.2. Heavy Quark Effective Theory*[14–24]

In particle physics, it is often the case that the effects of a very heavy particle become irrelevant at low energies. It is then useful to construct a low-energy effective theory, in which this heavy particle no longer appears. Eventually, this effective theory will be easier to deal with than the full theory. A familiar example is Fermi's theory of the weak interactions. For the description of weak decays of hadrons, one can approximate the weak interactions by point-like four-fermion couplings governed by a dimensionful coupling constant $G_F$. Only at energies much larger than the masses of hadrons can one resolve the structure of the intermediate vector bosons $W$ and $Z$.

Technically, the process of removing the degrees of freedom of a heavy particle involves the following steps[174–176]: One first identifies the heavy particle fields and "integrates them out" in the generating functional of the Green functions of the theory. This is possible since at low energies the heavy particle does not appear as an external state. However, although the action of the full theory is usually a local one, what results after this first step is a nonlocal effective action. The nonlocality is related to the fact that in the full theory the heavy particle (with mass $M$) can appear in virtual processes and propagate over a short but finite distance $\Delta x \sim 1/M$. Thus a second step is required to get to a local effective Lagrangian: The nonlocal effective action is rewritten as an infinite series of local terms in an operator product expansion (OPE)[177,178]. Roughly speaking, this corresponds to an expansion in powers of $1/M$. It is in this step that the short- and long-distance physics is disentangled. The long-distance physics corresponds to interactions at low energies and is the same in the full and the effective theory. But short-distance effects arising from quantum corrections involving large virtual momenta (of order $M$) are not reproduced in the effective theory, once the heavy particle has been integrated out. In a third step, they have to be added in a perturbative way using renormalization-group techniques. This procedure is called "matching". It leads to

renormalizations of the coefficients of the local operators in the effective Lagrangian. An example is the effective Lagrangian for nonleptonic weak decays, in which radiative corrections from hard gluons with virtual momenta in the range between $m_W$ and some renormalization scale $\mu \sim 1$ GeV give rise to Wilson coefficients, which renormalize the local four-fermion interactions[179–181].

The heavy quark effective theory (HQET) is constructed to provide a simplified description of processes where a heavy quark interacts with light degrees of freedom by the exchange of soft gluons. Clearly, $m_Q$ is the high-energy scale in this case, and $\Lambda_{\rm QCD}$ is the scale of the hadronic physics one is interested in. However, a subtlety arises since one wants to describe the properties and decays of hadrons which contain a heavy quark. Hence, it is not possible to remove the heavy quark completely from the effective theory. What is possible, however, is to integrate out the "small components" in the full heavy quark spinor, which describe fluctuations around the mass shell.

The starting point in the construction of the low-energy effective theory is the observation that a very heavy quark bound inside a hadron moves with essentially the hadron's velocity $v$ and is almost on-shell. Its momentum can be written as

$$p_Q^\mu = m_Q v^\mu + k^\mu \,, \tag{100}$$

where the components of the so-called residual momentum $k^\mu$ are much smaller than $m_Q$. Interactions of the heavy quark with light degrees of freedom change the residual momentum by an amount of order $\Delta k^\mu \sim \Lambda_{\rm QCD}$, but the corresponding changes in the heavy quark velocity vanish as $\Lambda_{\rm QCD}/m_Q \to 0$. In this situation, it is appropriate to introduce "large" and "small" component fields $h_v$ and $H_v$ by

$$\begin{aligned} h_v(x) &= e^{i m_Q v\cdot x}\, P_+\, Q(x) \,, \\ H_v(x) &= e^{i m_Q v\cdot x}\, P_-\, Q(x) \,, \end{aligned} \tag{101}$$

where $P_+$ and $P_-$ are projection operators defined as

$$P_\pm = \frac{1 \pm \not{v}}{2} \,. \tag{102}$$

It follows that

$$Q(x) = e^{-i m_Q v\cdot x} \Big[ h_v(x) + H_v(x) \Big] \,. \tag{103}$$

Because of the projection operators, the new fields satisfy $\not{v}\, h_v = h_v$ and $\not{v}\, H_v = -H_v$. In the rest frame, $h_v$ corresponds to the upper two components of $Q$, while $H_v$ corresponds to the lower ones. Whereas $h_v$ annihilates a heavy quark with velocity $v$, $H_v$ creates a heavy antiquark with velocity $v$. If the heavy quark was on-shell, the field $H_v$ would be absent.

In terms of the new fields, the QCD Lagrangian (99) for a heavy quark takes the following form:

$$\mathcal{L}_Q = \bar h_v\, iv\cdot D\, h_v - \bar H_v \,(iv\cdot D + 2m_Q)\, H_v + \bar h_v\, i \not{D}_\perp H_v + \bar H_v\, i \not{D}_\perp h_v \,, \tag{104}$$

where $D_\perp^\mu = D^\mu - v^\mu\, v{\cdot}D$ is orthogonal to the heavy quark velocity: $v{\cdot}D_\perp = 0$. In the rest frame, $D_\perp^\mu = (0, \vec{D}\,)$ contains the spatial components of the covariant derivative. From (104), it is apparent that $h_v$ describes massless degrees of freedom, whereas $H_v$ corresponds to fluctuations with twice the heavy quark mass. These are the heavy degrees of freedom, which will be eliminated in the construction of the effective theory. The fields are mixed by the presence of the third and fourth terms, which describe pair creation or annihilation of heavy quarks and antiquarks. As shown in the first diagram in Fig. 11, in a virtual process a heavy quark propagating forward in time can turn into a virtual antiquark propagating backward in time, and then turn back into a quark. The energy of the intermediate quantum state $hh\bar{H}$ is larger than the energy of the incoming heavy quark by at least $2m_Q$. Because of this large energy gap, the virtual quantum fluctuation can only propagate over a short distance $\Delta x \sim 1/m_Q$. On hadronic scales set by $R_{\rm had} = 1/\Lambda_{\rm QCD}$, the process essentially looks like a local interaction of the form

$$\bar{h}_v\, i \not{D}_\perp \frac{1}{2m_Q}\, i \not{D}_\perp h_v \,, \tag{105}$$

where I have simply replaced the propagator for $H_v$ by $1/2m_Q$. A more correct treatment is to integrate out the small component field $H_v$, thereby deriving a nonlocal effective action for the large component field $h_v$, which can then be expanded in terms of local operators. Before doing this, let me mention a second type of virtual corrections that involve pair creation, namely heavy quark loops. An example is depicted in the second diagram in Fig. 11. Heavy quark loops cannot be described in terms of the effective fields $h_v$ and $H_v$, since the quark velocities in the loop are not conserved, and are in no way related to hadron velocities. However, such short-distance processes are proportional to the small coupling constant $\alpha_s(m_Q)$ and can be calculated in perturbation theory. They lead to corrections which are added onto the low-energy effective theory in the matching procedure.

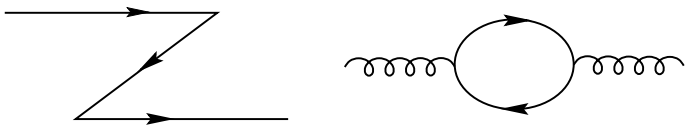

Figure 11: Virtual fluctuations involving pair creation of heavy quarks. Time flows to the right.

On a classical level, the heavy degrees of freedom represented by $H_v$ can be eliminated using the equations of motion of QCD. Substituting (103) into $(i \not{D} - m_Q)\, Q = 0$ gives

$$i \not{D}\, h_v + (i \not{D} - 2m_Q)\, H_v = 0 \,, \tag{106}$$

and multiplying this by $P_\pm$ one derives the two equations

$$\begin{aligned} -\, iv{\cdot}D\, h_v &= i \not{D}_\perp H_v \,, \\ (iv{\cdot}D + 2m_Q)\, H_v &= i \not{D}_\perp h_v \,. \end{aligned} \tag{107}$$

The second can be solved to give ($\eta \to +0$)

$$H_v = \frac{1}{(2m_Q + iv\cdot D - i\eta)}\, i\not{D}_\perp h_v\,, \tag{108}$$

which shows that the small component field $H_v$ is indeed of order $1/m_Q$. One can now insert this solution into the first equation to obtain the equation of motion for $h_v$. It is easy to see that this equation follows from the nonlocal effective Lagrangian

$$\mathcal{L}_{\rm eff} = \bar h_v\, iv\cdot D\, h_v + \bar h_v\, i\not{D}_\perp \frac{1}{(2m_Q + iv\cdot D - i\eta)}\, i\not{D}_\perp h_v\,. \tag{109}$$

Clearly, the second term precisely corresponds to the first class of virtual processes depicted in Fig. 11.

Mannel, Roberts, and Ryzak have derived this Lagrangian in a more elegant way by manipulating the generating functional for QCD Green's functions containing heavy quark fields[182]. They start from the field redefinition (103) and couple the large component fields $h_v$ to external sources $\rho_v$. Green's functions with an arbitrary number of $h_v$ fields can be constructed by taking derivatives with respect to $\rho_v$. No sources are needed for the heavy degrees of freedom represented by $H_v$. The functional integral over these fields is Gaussian and can be performed explicitly, leading to the nonlocal effective action

$$S_{\rm eff} = \int {\rm d}^4x\, \mathcal{L}_{\rm eff} - i\ln\Delta\,, \tag{110}$$

with $\mathcal{L}_{\rm eff}$ as given in (109). The appearance of the logarithm of the determinant

$$\Delta = \exp\left(\frac{1}{2}\,{\rm Tr}\,\ln\left[2m_Q + iv\cdot D - i\eta\right]\right) \tag{111}$$

is a quantum effect not present in the classical derivation given above. However, in this case the determinant can be regulated in a gauge-invariant way, and by choosing the axial gauge $v\cdot A = 0$ one shows that $\ln\Delta$ is just an irrelevant constant[182,183].

Because of the phase factor in (103), the $x$ dependence of the effective heavy quark field $h_v$ is weak. In momentum space, derivatives acting on $h_v$ produce powers of the residual momentum $k$, which is much smaller than $m_Q$. Hence, the nonlocal effective Lagrangian (109) allows for a derivative expansion in powers of $iD/m_Q$. Taking into account that $h_v$ contains a $P_+$ projection operator, and using the identity

$$P_+\, i\not{D}_\perp\, i\not{D}_\perp P_+ = P_+\left[(iD_\perp)^2 + \frac{g_s}{2}\,\sigma_{\alpha\beta}\,G^{\alpha\beta}\right]P_+\,, \tag{112}$$

where $[iD^\alpha, iD^\beta] = ig_s\,G^{\alpha\beta}$ is the gluon field-strength tensor, one finds that[19,23]

$$\mathcal{L}_{\rm eff} = \bar h_v\, iv\cdot D\, h_v + \frac{1}{2m_Q}\,\bar h_v\,(iD_\perp)^2\, h_v + \frac{g_s}{4m_Q}\,\bar h_v\,\sigma_{\alpha\beta}\,G^{\alpha\beta}\,h_v + O(1/m_Q^2)\,. \tag{113}$$

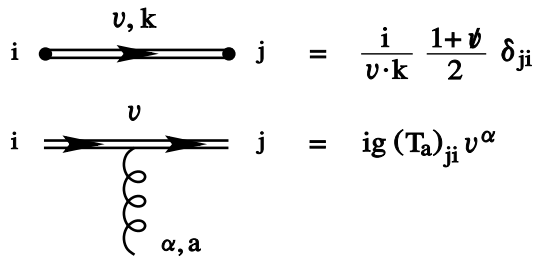


Figure 12: Feynman rules of the heavy quark effective theory ($i, j$ and $a$ are colour indices). It has become standard to represent a heavy quark by a double line. The residual momentum $k$ is defined in (100).

In the limit $m_Q \to \infty$, only the first terms remains:

$$\mathcal{L}_\infty = \bar{h}_v \, iv\cdot D \, h_v \,. \tag{114}$$

This is the effective Lagrangian of HQET. From it are derived the Feynman rules depicted in Fig. 12. Let me take a moment to study the symmetries of this Lagrangian[21]. Since there appear no Dirac matrices, interactions of the heavy quark with gluons leave its spin unchanged. Associated with this is an SU(2) symmetry group, under which $\mathcal{L}_\infty$ is invariant. The action of this symmetry on the heavy quark fields becomes most transparent in the rest frame, where the generators $S^i$ of spin SU(2) can be chosen as

$$S^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{1}{2} \, \gamma_5 \gamma^0 \gamma^i \,, \tag{115}$$

where $\sigma^i$ are the Pauli matrices. In a general frame, one defines a set of three orthonormal vectors $e^i$ orthogonal to $v$, and takes the generators of the spin symmetry as $S^i = \frac{1}{2} \gamma_5 \, \not{v} \not{e}^i$. These matrices satisfy the commutation relations of SU(2) and commute with $\not{v}$:

$$[S^i, S^j] = i\epsilon^{ijk} S^k \,, \qquad [\not{v}, S^i] = 0 \,. \tag{116}$$

An infinitesimal SU(2) transformation $h_v \to (1 + i\vec{\epsilon}\cdot\vec{S}\,) \, h_v$ leaves the Lagrangian invariant,

$$\delta\mathcal{L}_\infty = \bar{h}_v \, [iv\cdot D, i\vec{\epsilon}\cdot\vec{S}\,] \, h_v = 0 \,, \tag{117}$$

and preserves the on-shell condition $\not{v} \, h_v = h_v$, since

$$\not{v} \, (1 + i\vec{\epsilon}\cdot\vec{S}\,) \, h_v = (1 + i\vec{\epsilon}\cdot\vec{S}\,) \, \not{v} \, h_v = (1 + i\vec{\epsilon}\cdot\vec{S}\,) \, h_v \,. \tag{118}$$

There is another symmetry of HQET, which arises since the mass of the heavy quark does not appear in the effective Lagrangian. When there are $N_h$ heavy quarks

moving at the same velocity, one can simply extend (114) by writing

$$\mathcal{L}_\infty = \sum_{i=1}^{N_h} \bar{h}_v^i \, iv\cdot D \, h_v^i \,. \tag{119}$$

This is clearly invariant under rotations in flavour space. When combined with the spin symmetry, the symmetry group becomes promoted to SU($2N_h$). This is the heavy quark spin-flavour symmetry[13,21]. Its physical content is that, in the $m_Q \to \infty$ limit, the strong interactions of a heavy quark become independent of its mass and spin.

Let me now have a look at the operators appearing at order $1/m_Q$ in (113). They are easiest to identify in the rest frame. The first operator,

$$\mathcal{O}_{\rm kin} = \frac{1}{2m_Q} \, \bar{h}_v \, (iD_\perp)^2 \, h_v \to -\frac{1}{2m_Q} \, \bar{h}_v \, (i\vec{D}\,)^2 \, h_v \,, \tag{120}$$

is the gauge-covariant extension of the kinetic energy arising from the off-shell residual motion of the heavy quark. The second operator is the non-Abelian analogue of the Pauli term, which describes the colour-magnetic interaction of the heavy quark spin with the gluon field:

$$\mathcal{O}_{\rm mag} = \frac{g_s}{4m_Q} \, \bar{h}_v \, \sigma_{\alpha\beta} \, G^{\alpha\beta} \, h_v \to -\frac{g_s}{m_Q} \, \bar{h}_v \, \vec{S}\cdot\vec{B}_c \, h_v \,. \tag{121}$$

Here $\vec{S}$ is the spin operator defined in (115), and $B_c^i = -\frac{1}{2}\epsilon^{ijk} G^{jk}$ are the components of the colour-magnetic field. This chromo-magnetic interaction is a relativistic effect, which scales like $1/m_Q$. This is the origin of the heavy quark spin symmetry.

### *3.3. Spectroscopic Implications*

The spin-flavour symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states[185]. In the $m_Q \to \infty$ limit, the spin of the heavy quark and the total angular momentum $j$ of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavour, spin, parity, etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm \frac{1}{2}$. The flavour symmetry relates the properties of states with different heavy quark flavour.

Consider, as an example, the ground-state mesons containing a heavy quark. In this case the light degrees of freedom have the quantum numbers of an antiquark, and the degenerate states are the pseudoscalar ($J = 0$) and vector ($J = 1$) mesons. In the charm and bottom systems, one knows experimentally

$$\begin{aligned} m_{B^*} - m_B &\simeq 46 \text{ MeV} \,, \\ m_{D^*} - m_D &\simeq 142 \text{ MeV} \,. \end{aligned} \tag{122}$$

These mass splittings are in fact reasonably small. To be more specific, at order $1/m_Q$ one expects hyperfine corrections to resolve the degeneracy, for instance $m_{B^*} - m_B \propto 1/m_b$. This leads to the refined prediction $m_{B^*}^2 - m_B^2 \simeq m_{D^*}^2 - m_D^2 \simeq$ const. The data are compatible with this:

$$\begin{aligned} m_{B^*}^2 - m_B^2 &\simeq 0.49 \text{ GeV}^2 , \\ m_{D^*}^2 - m_D^2 &\simeq 0.55 \text{ GeV}^2 . \end{aligned} \tag{123}$$

One can also study excited meson states, in which the light constituents carry orbital angular momentum. It is tempting to interpret $D_1(2420)$ with $J^P = 1^+$ and $D_2(2460)$ with $J^P = 2^+$ as the spin doublet corresponding to $j = \frac{3}{2}$. The small mass difference $m_{D_2^*} - m_{D_1} \simeq 35$ MeV supports this assertion. One then expects

$$m_{B_2^*}^2 - m_{B_1}^2 \simeq m_{D_2^*}^2 - m_{D_1}^2 \simeq 0.17 \text{ GeV}^2 \tag{124}$$

for the corresponding states in the bottom system. The fact that this mass splitting is smaller than for the ground-state mesons is not unexpected. For instance, in the nonrelativistic constituent quark model the light antiquark in these excited mesons is in a $p$-wave state, and its wave function at the location of the heavy quark vanishes. Hence, in this model hyperfine corrections are strongly suppressed.

A typical prediction of the flavour symmetry is that the "excitation energies" for states with different quantum numbers of the light degrees of freedom are approximately the same in the charm and bottom systems. For instance, one expects

$$\begin{aligned} m_{B_S} - m_B &\simeq m_{D_S} - m_D \simeq 100 \text{ MeV} , \\ m_{B_1} - m_B &\simeq m_{D_1} - m_D \simeq 557 \text{ MeV} , \\ m_{B_2^*} - m_B &\simeq m_{D_2^*} - m_D \simeq 593 \text{ MeV} . \end{aligned} \tag{125}$$

The first relation has been very nicely confirmed by the discovery of the $B_S$ meson at LEP[186]. The observed mass[124] $m_{B_S} = 5.368 \pm 0.005$ GeV corresponds to an excitation energy $m_{B_S} - m_B = 89 \pm 5$ MeV.

### *3.4. Weak Decay Form Factors*

Of particular interest are the relations between the weak decay form factors of heavy mesons, which parametrize hadronic matrix elements of currents between two meson states containing a heavy quark. These relations have been derived by Isgur and Wise[13], generalizing ideas developed by Nussinov and Wetzel[10] and by Voloshin and Shifman[11,12]. For the purpose of this discussion, it is convenient to work with a mass-independent normalization of meson states,

$$\langle M(p')|M(p)\rangle = \frac{2p^0}{m_M} (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') , \tag{126}$$

instead of the more conventional, relativistic normalization

$$\langle \widetilde{M}(p')|\widetilde{M}(p)\rangle = 2p^0 (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') . \tag{127}$$

In the first case, $p^0/m_M = v^0$ depends only on the meson velocity. In fact, it is more natural for heavy quark systems to use velocity rather than momentum variables. I will thus write $|M(v)\rangle$ instead of $|M(p)\rangle$. The relation to the conventionally normalized states is $|M(v)\rangle = m_M^{-1/2}\,\widetilde{M}(p)\rangle$.

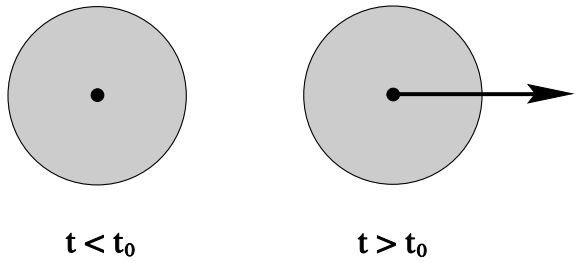


Figure 13: The action of an external heavy quark current, as seen by the light degrees of freedom in the initial state.

Consider now the elastic scattering of a pseudoscalar meson, $P(v) \to P(v')$, induced by an external vector current coupled to the heavy quark contained in $P$. Before the action of the current, the light degrees of freedom in the initial state orbit around the heavy quark, which acts as a source of colour moving with the meson's velocity $v$. On average, this is also the velocity of the "brown muck". The action of the current is to replace instantaneously (at $t = t_0$) the colour source by one moving at velocity $v'$, as indicated in Fig. 13. If $v = v'$, nothing really happens; the light degrees of freedom do not realize that there was a current acting on the heavy quark. If the velocities are different, however, the "brown muck" suddenly finds itself interacting with a moving (relative to its rest frame) colour source. Soft gluons have to be exchanged in order to rearrange the light degrees of freedom and build the final state meson moving at velocity $v'$. This rearrangement leads to a form factor suppression, which reflects the fact that as the velocities become more and more different, the probability for an elastic transition decreases. The important observation is that, in the $m_Q \to \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ that connects the rest frames of the initial- and final-state mesons. That the form factor is independent of the heavy quark mass can also be seen from the following intuitive argument: The light constituents in the initial and final state carry momenta that are typically of order $\Lambda_{\rm QCD} v$ and $\Lambda_{\rm QCD} v'$, respectively. Thus, their momentum transfer is $q^2 \sim \Lambda_{\rm QCD}^2 (v \cdot v' - 1)$, which is in fact independent of $m_Q$.

The result of this discussion is that in the effective theory, which provides the appropriate framework to consider the limit $m_Q \to \infty$ with the quark velocities kept fixed, the hadronic matrix element describing the scattering process can be written as[13]

$$\langle P(v')|\,\bar{h}_{v'} \gamma^\mu h_v\,|P(v)\rangle = \xi(v \cdot v')\,(v + v')^\mu\,, \tag{128}$$

with a form factor $\xi(v \cdot v')$ that does not depend on $m_Q$. Since the matrix element

is invariant under complex conjugation combined with an interchange of $v$ and $v'$, the function $\xi(v\cdot v')$ must be real. That there is no term proportional to $(v-v')^\mu$ can be seen by contracting the matrix element with $(v-v')_\mu$, and using $\not{v} h_v = h_v$ and $\bar h_{v'}\not{v}' = \bar h_{v'}$.

One can now use the flavour symmetry to replace the heavy quark $Q$ in one of the meson states by a heavy quark $Q'$ of a different flavour, thereby turning $P$ into another pseudoscalar meson $P'$. At the same time, the current becomes a flavour-changing vector current. In the infinite mass limit, this is a symmetry transformation, under which the effective Lagrangian is invariant. Hence, the matrix element

$$\langle P'(v')|\,\bar h'_{v'}\gamma^\mu h_v\,|P(v)\rangle = \xi(v\cdot v')\,(v+v')^\mu \tag{129}$$

is still determined by the same function $\xi(v\cdot v')$. This universal form factor is called the Isgur–Wise function, after the discoverers of this relation[13].

For equal velocities, the vector current $J^\mu = \bar h'_v\gamma^\mu h_v = \bar h'_v v^\mu h_v$ is conserved in the effective theory, irrespective of the flavour of the heavy quarks, since

$$i\partial_\mu J^\mu = \bar h'_v\,(iv\cdot D\,h_v) + (iv\cdot D\,\bar h'_v)\,h_v = 0 \tag{130}$$

by the equation of motion, $iv\cdot D\,h_v = 0$, which follows from the effective Lagrangian (114). The conserved charges

$$N_{Q'Q} = \int \mathrm{d}^3x\,J^0(x) = \int \mathrm{d}^3x\,h'^\dagger_v(x)\,h_v(x) \tag{131}$$

are generators of the flavour symmetry. The diagonal generators simply count the number of heavy quarks, whereas the off-diagonal ones replace a heavy quark by another: $N_{QQ}|P(v)\rangle = |P(v)\rangle$ and $N_{Q'Q}|P(v)\rangle = |P'(v)\rangle$. It follows that

$$\langle P'(v)|\,N_{Q'Q}\,|P(v)\rangle = \langle P(v)|P(v)\rangle = 2v^0\,(2\pi)^3\,\delta^3(\vec 0\,)\,, \tag{132}$$

and from a comparison with (129) one concludes that the Isgur–Wise function is normalized at the point of equal velocities:

$$\xi(1) = 1\,. \tag{133}$$

This can easily be understood in terms of the physical picture discussed above: When there is no velocity change, the light degrees of freedom see the same colour field and are in an identical configuration before and after the action of the current. There is no form factor suppression. Since $E_{\rm recoil} = m_{P'}\,(v\cdot v'-1)$ is the recoil energy of the daughter meson $P'$ in the rest frame of the parent meson $P$, the point $v\cdot v' = 1$ is referred to as the zero recoil limit.

The heavy quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons. In the effective theory, a vector meson with longitudinal polarization $\epsilon_3$ is related to a pseudoscalar meson by

$$|V(v,\epsilon_3)\rangle = 2\,\mathbf{S}^3_Q\,|P(v)\rangle\,, \tag{134}$$

where $\mathbf{S}_Q^3$ is a Hermitian operator that acts on the spin of the heavy quark $Q$. It follows that

$$\begin{aligned}\langle V'(v',\epsilon_3)|\,\bar h'_{v'}\Gamma\,h_v\,|P(v)\rangle &= \langle P'(v')|\,2\,[\mathbf{S}^3_{Q'},\bar h'_{v'}\Gamma\,h_v]\,|P(v)\rangle \\ &= \langle P'(v')|\,\bar h'_{v'}(2S^3\Gamma)\,h_v\,|P(v)\rangle\,, \end{aligned} \tag{135}$$

where $\Gamma$ can be an arbitrary combination of Dirac matrices, and $S^3$ is a matrix representation of the operator $\mathbf{S}^3_{Q'}$. It is easiest to evaluate this expression in the rest frame of the final-state meson, where [cf. (115)]

$$v'^\mu = (1,0,0,0)\,, \qquad \epsilon_3^\mu = (0,0,0,1)\,, \qquad S^3 = \frac{1}{2}\,\gamma_5\gamma^0\gamma^3\,. \tag{136}$$

It is then straightforward to obtain the following commutation relations for the components of the weak current $(V-A)^\mu = \bar h'_{v'}\gamma^\mu(1-\gamma_5)\,h_v$:

$$\begin{aligned} 2\,[\mathbf{S}^3_{Q'},V^0-A^0] &= A^3-V^3\,,\\ 2\,[\mathbf{S}^3_{Q'},V^3-A^3] &= A^0-V^0\,,\\ 2\,[\mathbf{S}^3_{Q'},V^1-A^1] &= \;\; i(A^2-V^2)\,,\\ 2\,[\mathbf{S}^3_{Q'},V^2-A^2] &= -i(A^1-V^1)\,. \end{aligned} \tag{137}$$

Using (135) and (137), one can relate the matrix element of the weak current between a pseudoscalar and a vector meson to the matrix element of the vector current between two pseudoscalar mesons given in (129). The result can be written in the covariant form[13]:

$$\begin{aligned}\langle V'(v',\epsilon)|\,\bar h'_{v'}\gamma^\mu(1-\gamma_5)\,h_v\,|P(v)\rangle &= i\epsilon^{\mu\nu\alpha\beta}\,\epsilon^*_\nu\,v'_\alpha v_\beta\;\xi(v\cdot v') \\ &\quad -\Big[\epsilon^{*\mu}\,(v\cdot v'+1)-v'^\mu\,\epsilon^*\cdot v\Big]\,\xi(v\cdot v')\,, \end{aligned} \tag{138}$$

where $\epsilon^{0123} = -\epsilon_{0123} = -1$. Once again, the matrix element is completely described in terms of the universal Isgur–Wise form factor.

Equations (129) and (138) summarize the relations imposed by heavy quark symmetry on the weak decay form factors describing the semileptonic decay processes $B\to D\,\ell\,\bar\nu$ and $B\to D^*\ell\,\bar\nu$. These relations are model-independent consequences of QCD in the limit where $m_b, m_c \gg \Lambda_{\rm QCD}$. They play a crucial role in the determination of $|\,V_{cb}|$, as I will discuss later in this lecture.

### *3.5. The $1/m_Q$ Expansion*

At tree level, eq. (113) defines the Lagrangian of HQET as a series of local, higher-dimension operators multiplied by powers of $1/m_Q$. The expression for $H_v$ in (108) can be used to derive a similar expansion for the full heavy quark field

$Q(x)$:

$$Q(x) = e^{-im_Q v\cdot x}\left[1 + \frac{1}{(2m_Q + iv\cdot D - i\eta)}\, i\not{D}_\perp\right] h_v(x)$$
$$= e^{-im_Q v\cdot x}\left(1 + \frac{i\not{D}_\perp}{2m_Q} + \dots\right) h_v(x)\,. \tag{139}$$

This relation contains the recipe for the construction (at tree level) of any operator in HQET that contains one or more heavy quark fields. For instance, the vector current $V^\mu = \bar{q}\,\gamma^\mu Q$ composed of a heavy and a light quark is represented as

$$V^\mu(x) = e^{-im_Q v\cdot x}\,\bar{q}(x)\,\gamma^\mu\left(1 + \frac{i\not{D}_\perp}{2m_Q} + \dots\right) h_v(x)\,. \tag{140}$$

Matrix elements of this current can be parametrized by hadronic form factors, and the purpose of using an effective theory is to make the $m_Q$ dependence of these form factors explicit.

Consider, as an example, the matrix element of $V^\mu(0)$ between a heavy meson $M(v)$ and some light final state $\ell$:

$$\langle\,\ell\,|\,V^\mu\,|M(v)\rangle = \langle\,\ell\,|\,\bar{q}\,\gamma^\mu h_v\,|M(v)\rangle + \frac{1}{2m_Q}\,\langle\,\ell\,|\,\bar{q}\,\gamma^\mu\, i\not{D}_\perp h_v\,|M(v)\rangle + \dots\,. \tag{141}$$

It would be nice if the matrix elements on the right-hand side of this equation were independent of $m_Q$. Then the second term would give the leading power correction to the first one. However, the equation of motion for $h_v$ derived from (113) contains $1/m_Q$ corrections, too. This leads to an $m_Q$ dependence of any hadronic matrix element of operators containing such fields. Another way to say this is that the eigenstates of the effective Lagrangian $\mathcal{L}_{\rm eff}$ (supplemented by the standard QCD Lagrangian for the light quarks and gluons) depend, at higher order in $1/m_Q$, on the heavy quark mass. This is no surprise, since the effective Lagrangian is equivalent to the Lagrangian of the full theory.

It is better to organize things in a slightly different way by working with the eigenstates of only the leading-order effective Lagrangian $\mathcal{L}_\infty$ in (114), treating the higher-dimension operators perturbatively as power corrections[24,184,187]. Then the equation of motion satisfied by $h_v$ is simply

$$iv\cdot D\, h_v = 0\,, \tag{142}$$

and the states of the effective theory are truly independent of $m_Q$. However, these states are now different from the states of the full theory. For instance, instead of (141) one should write

$$\langle\,\ell\,|\,V^\mu\,|M(v)\rangle_{\rm QCD} = \langle\,\ell\,|\,\bar{q}\,\gamma^\mu h_v\,|M(v)\rangle_\infty + \frac{1}{2m_Q}\,\langle\,\ell\,|\,\bar{q}\,\gamma^\mu\, i\not{D}_\perp h_v\,|M(v)\rangle_\infty$$

$$
\begin{aligned}
&+ \frac{1}{2m_Q} \langle\, \ell \,|\, i \int \mathrm{d}y \; T \left\{ \, \bar{q}\, \gamma^\mu h_v(0), \bar{h}_v \, (iD_\perp)^2 \, h_v(y) \, \right\} |M(v)\rangle_\infty \\
&+ \frac{g_s}{4m_Q} \langle\, \ell \,|\, i \int \mathrm{d}y \; T \left\{ \, \bar{q}\, \gamma^\mu h_v(0), \bar{h}_v \, \sigma_{\alpha\beta} \, G^{\alpha\beta} \, h_v(y) \, \right\} |M(v)\rangle_\infty \\
&+ O(1/m_Q^2) \,. \qquad (143)
\end{aligned}
$$

The matrix elements on the right-hand side are independent of the heavy quark mass. The mass dependence of the states $|M(v)\rangle_{\rm QCD}$ is reflected in HQET by the appearance of the third and fourth term, which arise from an insertion of the first-order power corrections in the Lagrangian into the leading-order matrix element of the current. This insertion can be thought of as being a correction to the wave function of the heavy meson. From now on I will omit the subscripts on the states. I will instead use the symbol "$\cong$" and write

$$
\begin{aligned}
V^\mu(0) \cong \bar{q}\, \gamma^\mu h_v(0) \,+\, & \frac{1}{2m_Q} \, \bar{q}\, \gamma^\mu \, i \not{D}_\perp h_v(0) \\
&+ \frac{1}{2m_Q} \, i \int \mathrm{d}y \; T \left\{ \, \bar{q}\, \gamma^\mu h_v(0), \bar{h}_v \, (iD_\perp)^2 \, h_v(y) \, \right\} \\
&+ \frac{g_s}{4m_Q} \, i \int \mathrm{d}y \; T \left\{ \, \bar{q}\, \gamma^\mu h_v(0), \bar{h}_v \, \sigma_{\alpha\beta} \, G^{\alpha\beta} \, h_v(y) \, \right\} + \ldots \qquad (144)
\end{aligned}
$$

to indicate that the operators on the two sides of this equation have to be evaluated between different states.

### *3.6. Matching and Running*

In Section 3.2, I have discussed the first two steps in the construction of HQET. Integrating out the small components in the heavy quark spinor fields, a nonlocal effective action was derived. This was then expanded in a series of local operators of increasing dimension to obtain an effective Lagrangian. A similar expansion could be written down for any external current. The effective Lagrangian and the effective currents derived that way correctly reproduce the long-distance physics of the full theory. They cannot describe the short-distance physics correctly, however. The reason is obvious: The heavy quark participates in strong interactions through its coupling to gluons. These gluons can be soft or hard, i.e. their virtual momenta can be small, of the order of the confinement scale, or large, of the order of the heavy quark mass. But hard gluons can resolve the nonlocality of the propagator of the small component fields $H_v$. Their effects are not taken into account in the naïve version of the OPE, which was used in the derivation of the effective Lagrangian in (113) and the effective vector current in (144). So far, the effective theory provides an appropriate description only at scales $\mu \ll m_Q$.

In this section, I will discuss the systematic treatment of short-distance corrections. A new feature of such corrections is that through the running coupling constant they induce a logarithmic dependence on the heavy quark mass[11]. The

important observation is that $\alpha_s(m_Q)$ is small, so that these effects can be calculated in perturbation theory. Consider, as an example, matrix elements of the vector current $V^\mu = \bar{q}\,\gamma^\mu Q$. In QCD this current is partially conserved and needs no renormalization[188]. Its matrix elements are free of ultraviolet divergences. Still, these matrix elements can have logarithmic dependence on $m_Q$ from the exchange of hard gluons with virtual momenta of the order of the heavy quark mass. If one goes over to the effective theory by taking the limit $m_Q \to \infty$, these logarithms diverge. Consequently, the vector current in the effective theory does require a renormalization[18]. Its matrix elements depend on an arbitrary renormalization scale $\mu$, which separates the regions of short- and long-distance physics. If $\mu$ is chosen such that $\Lambda_{\rm QCD} \ll \mu \ll m_Q$, the effective coupling constant in the region between $\mu$ and $m_Q$ is small, and perturbation theory can be used to compute the short-distance corrections. These corrections have to be added to the matrix elements of the effective theory, which only contain the long-distance physics below the scale $\mu$. Schematically, then, the relation between matrix elements in the full and in the effective theory is

$$\langle\, V^\mu(m_Q)\,\rangle_{\rm QCD} = C_0(m_Q,\mu)\,\langle V_0(\mu)\rangle_\infty + \frac{C_1(m_Q,\mu)}{2m_Q}\,\langle V_1(\mu)\rangle_\infty + \ldots\,, \tag{145}$$

where I have indicated that matrix elements of $V^\mu$ in the full theory depend on $m_Q$, whereas matrix elements of operators in the effective theory are mass-independent, but do depend on the renormalization scale. The Wilson coefficients $C_i(m_Q,\mu)$ are defined by this relation. Order by order in perturbation theory, they can be computed from a comparison of the matrix elements in both theories. Since the effective theory is constructed to reproduce correctly the low-energy behaviour of the full theory, this "matching" procedure is independent of any long-distance physics, such as infrared singularities, nonperturbative effects, the nature of the external states used in the matrix elements, physical cuts, etc. Only at high energies do the two theories differ, and these differences are corrected for by the short-distance coefficients.

The calculation of the coefficient functions in perturbation theory uses the powerful methods of the renormalization group, which I have discussed in my first lecture. It is in principle straightforward, yet in practice rather tedious. Much of the recent work on heavy quark symmetry has been devoted to this subject. In this section, I will discuss as an illustration the wave-function renormalization of the heavy quark field in HQET, and the short-distance expansion of a heavy–light vector current to leading order in $1/m_Q$. For a more comprehensive presentation of short-distance corrections, I refer to my review article[25].

In quantum field theory, the parameters and fields of the Lagrangian have no direct physical significance. They have to be renormalized before they can be related to observable quantities. In an intermediate step the theory has to be regularized. The most convenient regularization scheme in QCD is dimensional regularization[28,32,33], in which the dimension of space-time is analytically continued

to $D = 4 - 2\epsilon$, with $\epsilon$ being infinitesimal. Loop integrals that are logarithmically divergent in four dimensions become finite for $\epsilon > 0$. From the fact that the action $S = \int \mathrm{d}^D x\, \mathcal{L}(x)$ is dimensionless, one can derive the mass dimensions of the fields and parameters of the theory. For instance, one finds that the "bare" coupling constant $\alpha_s^{\rm bare}$ is no longer dimensionless if $D \neq 4$: $\dim[\,\alpha_s^{\rm bare}\,] = 2\epsilon$. In a renormalizable theory, it is possible to rewrite the Lagrangian in terms of renormalized quantities, in such a way that Green's functions of the renormalized fields remain finite as $\epsilon \to 0$. For QCD, one introduces renormalized quantities by $Q^{\rm bare} = Z_Q^{1/2}\, Q^{\rm ren}$, $A^{\rm bare} = Z_A^{1/2} A^{\rm ren}$, $\alpha_s^{\rm bare} = \mu^{2\epsilon} Z_\alpha\, \alpha_s^{\rm ren}$, etc., where $\mu$ is an arbitrary mass scale introduced to render the renormalized coupling constant dimensionless. Similarly, in HQET one defines a renormalized heavy quark field by $h_v^{\rm bare} = Z_h^{1/2}\, h_v^{\rm ren}$. From now on, the superscript "ren" will be omitted.

As a warm-up, let me sketch the calculation of the wave-function renormalization for a heavy quark in HQET. In the $\overline{\rm MS}$ scheme, $Z_h$ can be computed from the $1/\hat{\epsilon}$-pole in the quark self-energy shown in Fig. 14:

$$1 - Z_h^{-1} = \frac{1}{\hat{\epsilon}}\text{-pole of } \frac{\partial\Sigma(v\cdot k)}{\partial v\cdot k}\,. \tag{146}$$

As long as $v \cdot k < 0$, the self-energy is infrared finite and real. The result is gauge-dependent, however. In Feynman gauge, one obtains at one-loop order

$$\begin{aligned}
\Sigma(v\cdot k) &= -\frac{4ig_s^2}{3}\int\frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{(t^2+i\eta)[v\cdot(t+k)+i\eta]} \\
&= -\frac{8ig_s^2}{3}\int\limits_0^\infty \mathrm{d}\rho\int\frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{[t^2+2\rho\, v\cdot(t+k)+i\eta]^2} \\
&= \frac{2\alpha_s}{3\pi}\,\Gamma(\epsilon)\int\limits_0^\infty \mathrm{d}\rho\left(\frac{\rho^2+\rho\omega}{4\pi\mu^2}\right)^{-\epsilon},
\end{aligned} \tag{147}$$

where $\rho$ is a dimensionful Feynman parameter, and $\omega = -2v\cdot k > 0$ acts as an infrared cutoff. A straightforward calculation leads to

$$\begin{aligned}
\frac{\partial\Sigma(v\cdot k)}{\partial v\cdot k} &= \frac{4\alpha_s}{3\pi}\,\Gamma(1+\epsilon)\left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon}\int\limits_0^1 \mathrm{d}z\, z^{-1+2\epsilon}\,(1-z)^{-\epsilon} \\
&= \frac{4\alpha_s}{3\pi}\,\Gamma(2\epsilon)\,\Gamma(1-\epsilon)\left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon},
\end{aligned} \tag{148}$$

where I have substituted $\rho = \omega\,(1-z)/z$. From an expansion around $\epsilon = 0$, one obtains

$$Z_h = 1 + \frac{2\alpha_s}{3\pi\hat{\epsilon}}\,. \tag{149}$$

This result was first derived by Politzer and Wise[18]. In the meantime, the calculation was also done at the two-loop order[189–191].

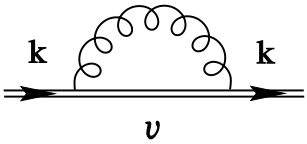


Figure 14: Self-energy $-i\Sigma(v\cdot k)$ of a heavy quark in HQET. The velocity $v$ is conserved by the strong interactions.

Similar to the fields and coupling constants, any composite operator built from quark and gluon fields may require a renormalization beyond that of its component fields. Such operators can be divided into three classes: gauge-invariant operators that do not vanish by the equations of motion (class-I), gauge-invariant operators that vanish by the equations of motion (class-II), and operators which are not gauge-invariant (class-III). In general, operators with the same dimension and quantum numbers mix under renormalization. However, things simplify if one works with the background field technique[192−195], which serves for quantizing gauge theories, preserving explicit gauge invariance. This offers the advantage that a class-I operator cannot mix with class-III operators, so that only gauge-invariant operators need be considered[196]. Furthermore, class-II operators are irrelevant since their matrix elements vanish by the equations of motion. It it thus sufficient to consider class-I operators only. For a set $\{\mathcal{O}_i\}$ of $n$ class-I operators that mix under renormalization, one defines an $n\times n$ matrix of renormalization factors $Z_{ij}$ by (summation over $j$ is understood) $\mathcal{O}_i^{\rm bare} = Z_{ij}\,\mathcal{O}_j$, such that the matrix elements of the renormalized operators $\mathcal{O}_j$ remain finite as $\epsilon\to 0$. In contrast to the bare operators, the renormalized ones depend on the subtraction scale via the $\mu$ dependence of $Z_{ij}$:

$$\mu\frac{\rm d}{{\rm d}\mu}\,\mathcal{O}_i = \left(\mu\frac{\rm d}{{\rm d}\mu}Z_{ij}^{-1}\right)\mathcal{O}_j^{\rm bare} = -\gamma_{ik}\,\mathcal{O}_k\,, \tag{150}$$

where

$$\gamma_{ik} = Z_{ij}^{-1}\,\mu\frac{\rm d}{{\rm d}\mu}\,Z_{jk} \tag{151}$$

are called the anomalous dimensions. It is sometimes convenient to introduce a compact matrix notation, in which $\vec{\mathcal{O}}$ is the vector of renormalized operators, $\hat{Z}$ is the matrix of renormalization factors, and $\hat{\gamma}$ denotes the anomalous dimension matrix. Then the running of the renormalized operators is controlled by the renormalization-group equation (RGE)

$$\left(\mu\frac{\rm d}{{\rm d}\mu}+\hat{\gamma}\right)\vec{\mathcal{O}} = 0\,. \tag{152}$$

As an example, consider the short-distance expansion of the heavy–light vector current $V^\mu = \bar{q}\,\gamma^\mu Q$ to leading order in $1/m_Q$. In the full theory, the vector current is not renormalized. But the effective current operators in HQET do require

renormalization. The general form of the short-distance expansion reads

$$\begin{aligned} V^\mu(m_Q) &\cong C_i(m_Q,\mu)\,\mathcal{O}_i(\mu) + O(1/m_Q) \\ &= C_i(m_Q,\mu)\,Z_{ij}^{-1}(\mu)\,\mathcal{O}_j^{\rm bare} + O(1/m_Q)\,, \end{aligned} \tag{153}$$

where I have indicated the $\mu$ dependence of the renormalized operators. This is the correct generalization of (145) in the case of operator mixing. In general, a complete set of operators with the same quantum numbers appears on the right-hand side. In the case at hand, there are two such operators, namely

$$\mathcal{O}_1 = \bar q\,\gamma^\mu h_v\,, \qquad \mathcal{O}_2 = \bar q\,v^\mu h_v\,. \tag{154}$$

From (152) and the fact that the product $C_i(m_Q,\mu)\,\mathcal{O}_i(\mu)$ must be $\mu$-independent, one can derive the RGE satisfied by the coefficient functions. It reads

$$\left(\mu\frac{\rm d}{{\rm d}\mu} - \hat\gamma^t\right)\vec C(m_Q,\mu) = 0\,, \tag{155}$$

where $\hat\gamma^t$ denotes the transposed anomalous dimension matrix, and I have collected the coefficients into a vector. The solution of the RGE proceeds as I described in my first lecture; however, it is technically more complicated in the case when there is operator mixing. One obtains

$$\vec C(m_Q,\mu) = \hat U(\mu,m_Q)\,\vec C(m_Q,m_Q)\,, \tag{156}$$

with the evolution matrix[27,43,44]

$$\hat U(\mu,m_Q) = T_\alpha\,\exp\int\limits_{\alpha_s(m_Q)}^{\alpha_s(\mu)} {\rm d}\alpha\,\frac{\hat\gamma^t(\alpha)}{\beta(\alpha)}\,. \tag{157}$$

Here "$T_\alpha$" means an ordering in the coupling constant such that the couplings increase from right to left (for $\mu < m_Q$). This is necessary since, in general, the anomalous dimension matrices at different values of $\alpha$ do not commute: $[\hat\gamma(\alpha_1),\hat\gamma(\alpha_2)] \neq 0$. In the present case, however, these complications are absent, since the anomalous dimension matrix turns out to be proportional to the unit matrix.

I will now illustrate the solution of (156) to next-to-leading order in renormalization-group improved perturbation theory. There are two ingredients to this calculation. First, the Wilson coefficients at the high-energy scale $\mu = m_Q$ must be calculated to one-loop order,

$$\vec C(m_Q,m_Q) = \vec C_{(0)} + \vec C_{(1)}\,\frac{\alpha_s(m_Q)}{4\pi} + \dots\,, \tag{158}$$

by matching QCD onto the effective theory. Then, in order to control the running of the HQET operators, the anomalous dimension matrix $\hat\gamma$ must be calculated to two-loop accuracy. Let me start with a comparison of the one-loop matrix elements of the

vector current in the full and in the effective theory. It is legitimate to perform the matching calculation with on-shell quark states. Then the matrix elements can be written as $\langle V^\mu \rangle = \bar{u}_q \, \Gamma^\mu \, u_Q$, where the heavy quark spinor satisfies $\not{v} \, u_Q = u_Q$. There will be infrared divergences in the calculation, which I regulate by introducing a fictitious gluon mass $\lambda$. At tree level, the vertex function in both theories is simply $\Gamma^\mu = \gamma^\mu$. The one-particle irreducible one-loop diagrams are shown in Fig. 15. They have to be supplemented by an on-shell wave-function renormalization of the external lines. The complete one-loop vertex function in QCD is[25]

$$\Gamma^\mu_{\rm QCD} = \left\{ 1 + \frac{\alpha_s}{2\pi} \left( \ln \frac{m_Q^2}{\lambda^2} - \frac{11}{6} \right) \right\} \gamma^\mu + \frac{2\alpha_s}{3\pi} \, v^\mu \,. \tag{159}$$

As required by the nonrenormalization theorem for partially conserved currents, this result is gauge-independent and ultraviolet finite[188]. In the limit of degenerate quark masses, the vector is conserved. The space integral over its time component is the generator of a flavour symmetry. It cannot be renormalized. When the symmetry is softly broken by the presence of mass splittings, this is only relevant for small loop momenta but does not affect the ultraviolet region. Thus, there can only be a finite renormalization. Notice, however, that (159) does contain an infrared singularity, because the matrix element was calculated using unphysical states.

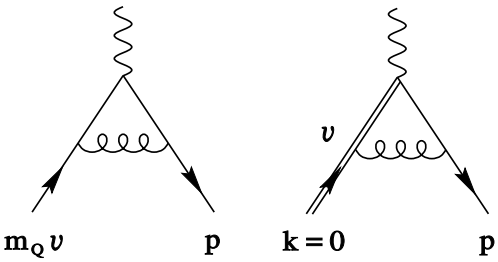


Figure 15: One-loop corrections to the matrix elements of the vector current in QCD and in HQET. The wavy line represents the current. The external momenta are on-shell.

In the effective theory, the one-loop vertex function for any heavy–light current $\bar{q} \, \Gamma \, h_v$ is found to be[25]

$$\left\{ 1 + \frac{\alpha_s}{2\pi} \left( \frac{1}{\hat{\epsilon}} + \ln \frac{\mu^2}{\lambda^2} + \frac{5}{6} \right) \right\} \Gamma \,, \tag{160}$$

where the Dirac structure of $\Gamma$ is completely arbitrary. It is generally true that these operators are renormalized multiplicatively and irrespectively of their Dirac structure[11]. Since there is no approximate flavour symmetry relating light and heavy quarks in the effective theory, it is not unexpected that the matrix elements of the bare currents are ultraviolet divergent. In the $\overline{\rm MS}$ scheme, the currents are renormalized by a diagonal matrix $\hat{Z}$ with components

$$Z_{11} = Z_{22} = 1 + \frac{\alpha_s}{2\pi\hat{\epsilon}} \,, \qquad Z_{12} = Z_{21} = 0 \,. \tag{161}$$

From (153), it then follows that the renormalized one-loop vertex function is

$$\Gamma^\mu_{\rm HQET} = \left\{1 + \frac{\alpha_s}{2\pi}\left(\ln\frac{\mu^2}{\lambda^2} + \frac{5}{6}\right)\right\}\left[C_1(m_Q,\mu)\,\gamma^\mu + C_2(m_Q,\mu)\,v^\mu\right]. \tag{162}$$

Notice that the $\lambda$ dependence is the same as in (159). The short-distance coefficients, which follow from a comparison of the two results, are independent of the infrared regulator:

$$\begin{aligned} C_1(m_Q,\mu) &= 1 + \frac{\alpha_s}{\pi}\left(\ln\frac{m_Q}{\mu} - \frac{4}{3}\right), \\ C_2(m_Q,\mu) &= \frac{2\alpha_s}{3\pi}. \end{aligned} \tag{163}$$

In particular, one obtains for the matching conditions at $\mu = m_Q$:

$$\begin{aligned} C_1(m_Q,m_Q) &= 1 - \frac{4\alpha_s(m_Q)}{3\pi}, \\ C_2(m_Q,m_Q) &= \frac{2\alpha_s(m_Q)}{3\pi}. \end{aligned} \tag{164}$$

This completes the first step of the calculation.

The next step is to calculate the one- and two-loop coefficients of the anomalous dimension, as defined in (13). From (160), it follows that the anomalous dimension matrix is proportional to the unit matrix, with the one-loop coefficient

$$\gamma_0 = -4. \tag{165}$$

This is the so-called hybrid anomalous dimension of heavy–light currents derived by Voloshin and Shifman[11]. In the context of the effective theory, it has been calculated by Politzer and Wise[18]. The two-loop coefficient $\gamma_1$ has been calculated by Ji and Musolf[189], and by Broadhurst and Grozin[190]. They obtain

$$\gamma_1 = -\frac{254}{9} - \frac{56}{27}\,\pi^2 + \frac{20}{9}\,n_f. \tag{166}$$

The final result for the Wilson coefficients can be derived using (20). It is possible to write the answer in the factorized form $C_i(m_Q,\mu) = \widehat{C}_i(m_Q)\,K_{\rm hl}(\mu)$, where the dependence on the two scales has been completely separated. One finds

$$\begin{aligned} \widehat{C}_1(m_Q) &= \Big[\alpha_s(m_Q)\Big]^{-2/\beta_0}\left\{1 + \frac{\alpha_s(m_Q)}{\pi}\left(S - \frac{4}{3}\right)\right\}, \\ \widehat{C}_2(m_Q) &= \Big[\alpha_s(m_Q)\Big]^{-2/\beta_0}\,\frac{2\alpha_s(m_Q)}{3\pi}, \\ K_{\rm hl}(\mu) &= \Big[\alpha_s(\mu)\Big]^{2/\beta_0}\left\{1 - \frac{\alpha_s(\mu)}{\pi}\,S\right\}, \end{aligned} \tag{167}$$

where

$$S = \frac{\gamma_1\beta_0 - \beta_1\gamma_0}{8\beta_0^2} = 3\,\frac{153 - 19n_f}{(33-2n_f)^2} - \frac{381 + 28\pi^2 - 30n_f}{36\,(33-2n_f)}\,. \tag{168}$$

*3.7. Flavour-Changing Heavy Quark Currents*

The OPE of currents in the effective theory becomes considerably more complicated in the case of flavour-changing currents. The weak current $\bar{c}\,\gamma^\mu(1-\gamma_5)\,b$ is of this form, however, so it is necessary to consider this case in detail. The complications arise from the fact that there are two different heavy quark masses, $m_b$ and $m_c$. Thus the calculation of the Wilson coefficients becomes a two-scale problem. In addition, the coefficients will depend in a nontrivial way on the velocity transfer $w = v\cdot v'$.

There are two obvious ways of performing the transition from QCD to an effective low-energy theory in which both the bottom and the charm quarks are treated as heavy quarks (in the sense of HQET): The transition can either be done in a single step, or by considering first an intermediate theory with a static bottom quark, but a dynamical charm quark. The latter becomes heavy in a second step. If one could solve perturbation theory to all orders, both treatments would lead to the same results for the Wilson coefficients. The calculation differs in both cases, however, and the results also differ if the perturbation series is truncated.

To see what the differences are, suppose first the two heavy quarks have similar masses, i.e. $m_b \sim m_c \sim m$, with $m$ being some average mass. It is then natural to remove at the same time the dynamical degrees of freedom of both heavy quarks. Let me consider the cases of the vector current $V^\mu = \bar{c}\,\gamma^\mu b$ and of the axial vector current $A^\mu = \bar{c}\,\gamma^\mu\gamma_5\,b$ explicitly and denote them collectively by $J^\mu$. Each of these currents obeys an expansion of the form

$$J^\mu(m_b,m_c) \cong \sum_{j=1}^{3} C_j(m_b,m_c,m,w,\mu)\,J_j(\mu) + O(1/m)\,, \tag{169}$$

where $J_j = \bar{h}^c_{v'}\,\Gamma_j h^b_v$ are local current operators in the effective theory with matrices

$$\Gamma_1 = \gamma^\mu\,, \qquad \Gamma_2 = v^\mu\,, \qquad \Gamma_3 = v'^\mu \tag{170}$$

for the vector current, and

$$\Gamma_1 = \gamma^\mu\gamma_5\,, \qquad \Gamma_2 = v^\mu\gamma_5\,, \qquad \Gamma_3 = v'^\mu\gamma_5 \tag{171}$$

for the axial vector current. In the above expression, I have indicated that matrix elements of the original current $J^\mu$ between states of the full theory depend on $m_b$ and $m_c$, whereas matrix elements of the operators $J_j$ between states of the effective theory are mass-independent, but do depend on the renormalization scale $\mu$. The short-distance coefficients are functions of the heavy quark masses, the

renormalization scale, and the matching scale $m$. The dependence on $m$ would disappear if one could sum the perturbative series to all orders. The advantage of this first approach is its simplicity. At leading order in the $1/m$ expansion, only three current operators contribute. Matrix elements of higher-dimension operators are suppressed by powers of $\Lambda_{\rm QCD}/m$. The short-distance coefficients contain the dependence on $m_b$ and $m_c$ correctly to a given order in $\alpha_s$, via matching at $\mu = m$. The only disadvantage is the residual dependence on the matching scale $m$, which arises when one calculates to finite order in perturbation theory. Although in a next-to-leading order calculation the scale in the leading anomalous scaling factor is determined, one has no control over the scale in the next-to-leading corrections proportional to $\alpha_s(m)$. Since $m_b > m > m_c$, this introduces an uncertainty of order $\alpha_s^2 \ln(m_b/m_c)$.

The alternative approach is to consider first, as an intermediate step for $m_b > \mu > m_c$, an effective theory with only a heavy bottom quark. Denoting the effective current operators of dimension $(3+k)$ in this theory by $\widetilde{J}_i^{(k)}(m_c,\mu)$, indicating that their matrix elements depend both on the charm quark mass and the renormalization scale, I write the short-distance expansion as an expansion in $1/m_b$:

$$J^\mu(m_b,m_c) \cong \sum_{k=0}^{\infty} \frac{1}{m_b^k} \sum_i D_i^{(k)}(m_b,\mu)\, \widetilde{J}_i^{(k)}(m_c,\mu)\,. \tag{172}$$

Since the velocity of the charm quark is still a dynamical degree of freedom, in this intermediate effective theory there are only two dimension-three operators, namely

$$\widetilde{J}_1^{(0)} = \bar c\,\Gamma_1 h_v^b\,, \qquad \widetilde{J}_2^{(0)} = \bar c\,\Gamma_2 h_v^b\,. \tag{173}$$

A major complication arises from the fact that matrix elements of the higher-dimension operators $\widetilde{J}_i^{(k)}$ will, in general, scale like $m_c^k$. This compensates the prefactor $1/m_b^k$. Consequently, these operators cannot be neglected even at leading order in the heavy quark expansion. One would thus have to deal with an infinite number of operators in order to keep track of the full dependence on the heavy quark masses. Ignoring this difficulty for the moment, one may use (172) to scale the currents from $\mu = m_b$ down to $\mu = m_c$, where the matching is done onto the final effective theory with two heavy quarks. In this step, the operator basis collapses considerably. When terms of order $\Lambda_{\rm QCD}/m_Q$ (here I use $m_Q$ generically for $m_c$ or $m_b$) are neglected on the level of matrix elements, only the three operators $J_j(\mu)$ in (169) remain. Each of the operators of the intermediate theory has an expansion in terms of these operators, with coefficients $E_{ij}$:

$$\widetilde{J}_i^{(k)}(m_c,m_c) \cong \sum_{j=1}^{3} m_c^k \left\{ E_{ij}^{(k)}(m_c,w,\mu)\, J_j(\mu) + O(1/m_c) \right\}. \tag{174}$$

Combining this with (172), one obtains

$$J^\mu(m_b,m_c) \cong \sum_{j=1}^{3} C_j(m_b,m_c,w,\mu)\, J_j(\mu) + O(1/m_Q)\,, \tag{175}$$

with evolution coefficients[197]

$$C_j(m_b, m_c, w, \mu) = \sum_{k=0}^{\infty} \Big(\frac{m_c}{m_b}\Big)^k \sum_i D_i^{(k)}(m_b, m_c)\, E_{ij}^{(k)}(m_c, w, \mu)\,. \tag{176}$$

In this expression, the matching scale $m$ of the first approach does not appear. The coefficients depend either on $\alpha_s(m_b)$ or $\alpha_s(m_c)$, i.e. the scaling in the intermediate region $m_b > \mu > m_c$ is properly taken into account. To achieve this, however, it would be necessary to consider an infinite number of operators in the intermediate effective theory. This is, of course, not manageable. It is important to realize, however, that the short-distance coefficients in (169) and (175) must agree, and that this equality must hold order by order in an expansion in the mass ratio $m_c/m_b$. Using this fact, one can combine the two approaches into a consistent next-to-leading order calculation[197]. For details of the very tedious calculation, the reader is referred to the literature[22,197–201]. Below, I will only give numerical results.

The short-distance coefficients can be written in the factorized form

$$C_i^{(5)}(m_b, m_c, w, \mu) = \widehat{C}_i^{(5)}(m_b, m_c, w)\, K_{\rm hh}(w, \mu)\,. \tag{177}$$

I denote the coefficients for the vector current by $\widehat{C}_i$, and those for the axial vector current by $\widehat{C}_i^5$. The $\mu$-dependent function $K_{\rm hh}(w, \mu)$ is universal and normalized to unity at zero recoil: $K_{\rm hh}(1, \mu) = 1$. The renormalization-group invariant coefficients $\widehat{C}_i^{(5)}$ contain all dependence on the heavy quark masses. In Table 1, I give the numerical values of these coefficients, which are obtained using $m_b = 4.80$ GeV and $m_c = 1.45$ GeV for the heavy quark masses, as well as $\Lambda_{\overline{\rm MS}} = 0.25$ GeV (for $n_f = 4$) in the two-loop expression for $\alpha_s(\mu)$. The corresponding coupling constants are $\alpha_s(m_b) \simeq 0.20$ and $\alpha_s(m_c) \simeq 0.32$.

Table 1: Short-distance coefficients for $b \to c$ transitions.

| $w$ | $\widehat{C}_1$ | $\widehat{C}_2$ | $\widehat{C}_3$ | $\widehat{C}_1^5$ | $\widehat{C}_2^5$ | $\widehat{C}_3^5$ |
|---|---|---|---|---|---|---|
| 1.0 | 1.136 | −0.085 | −0.021 | 0.985 | −0.122 | 0.042 |
| 1.1 | 1.107 | −0.080 | −0.021 | 0.965 | −0.115 | 0.040 |
| 1.2 | 1.081 | −0.077 | −0.020 | 0.946 | −0.109 | 0.038 |
| 1.3 | 1.056 | −0.073 | −0.019 | 0.927 | −0.103 | 0.036 |
| 1.4 | 1.033 | −0.070 | −0.018 | 0.910 | −0.098 | 0.035 |
| 1.5 | 1.011 | −0.067 | −0.018 | 0.894 | −0.094 | 0.033 |
| 1.6 | 0.991 | −0.064 | −0.017 | 0.878 | −0.089 | 0.032 |
| 1.7 | 0.972 | −0.062 | −0.017 | 0.864 | −0.086 | 0.031 |
| 1.8 | 0.953 | −0.059 | −0.016 | 0.850 | −0.082 | 0.030 |

Of particular interest for the model-independent determination of $|V_{cb}|$ to be discussed below is the value of the short-distance coefficient $\widehat{C}_1^5(w)$ at zero recoil.

One finds[197]

$$\eta_A \equiv \widehat{C}_1^5(1) = x^{6/25}\left\{1 + 1.561\,\frac{\alpha_s(m_c) - \alpha_s(m_b)}{\pi} - \frac{\alpha_s(m_c)}{\pi}\left(\frac{8}{3} + \frac{2z^2}{1-z}\,\ln z\right)\right.$$
$$\left. + z\left(\frac{25}{54} - \frac{14}{27}\,x^{-9/25} + \frac{1}{18}\,x^{-12/25} + \frac{8}{25}\,\ln x\right)\right\}, \qquad (178)$$

where $x = \alpha_s(m_c)/\alpha_s(m_b)$, and $z = m_c/m_b$. For $\Lambda_{\overline{\rm MS}} = 0.25 \pm 0.05$ GeV and $z = 0.30 \pm 0.05$, this yields $\eta_A = 0.986 \pm 0.006$.

Let me stop, at this point, the discussion of short-distance effects. There have been important calculations that I have no time to mention here, such as the renormalization of the higher-dimension operators in the effective Lagrangian[19,23,202], and the short-distance expansion of currents to next-to-leading order in[203] $1/m_Q$. A discussion of these calculations can be found in my review article[25].

### *3.8. Covariant Representation of States*

The purpose of the OPE discussed in the previous sections was to disentangle the short-distance physics related to length scales set by the Compton wavelengths of the heavy quarks from confinement effects relevant at large distances. This procedure makes explicit the $m_Q$ dependence of any Green's function of the full theory, which contains one or more heavy quark fields. Hadronic matrix elements of heavy quark currents have a $1/m_Q$ expansion as shown in (145). The HQET matrix elements on the right-hand side of this equation contain the long-distance physics associated with the interactions of the cloud of light degrees of freedom among themselves and with the background colour-field provided by the heavy quarks. These hadronic quantities depend in a most complicated way on the "brown muck" quantum numbers of the external states, the quantum numbers of the current, and on the heavy quark velocities. They are related to matrix elements of effective current operators in HQET. Recall that these matrix elements are independent of the heavy quark masses, if the states of the effective theory are taken to be the eigenstates of the leading-order Lagrangian $\mathcal{L}_\infty$ in (114). Matrix elements evaluated using these states have a well-defined behaviour under spin-flavour symmetry transformations. When combined with the requirements of Lorentz covariance, restrictive constraints on their structure can be derived. I will now introduce a very elegant formalism due to Bjorken[204] and Falk et al.[22,205], which allows one to derive the general form of such matrix elements in a straightforward manner. The clue is to work with a covariant tensor representation of states with definite transformation properties under the Lorentz group and the heavy quark spin-flavour symmetry.

The eigenstates of HQET can be thought of as the "would-be hadrons" built from an infinitely heavy quark dressed with light quarks, antiquarks and gluons. In such a state, both the heavy quark and the cloud of light degrees of freedom have well-defined transformation properties under the Lorentz group. The heavy quark can be represented by a spinor $u_h(v,s)$ satisfying $\not{v}\,u_h(v,s) = u_h(v,s)$, where $v$ is the velocity of the hadron. Because of heavy quark symmetry, the wave function

of the state (when properly normalized) is independent of the flavour and spin of the heavy quark, and the states can be characterized by the quantum numbers of the "brown muck". In particular, for each configuration of light degrees of freedom with total angular momentum $j \geq 0$ and parity $P$, there is a degenerate doublet of states with spin-parity $J^P = (j \pm \frac{1}{2})^P$. Following Falk[205], I discuss the cases of integral and half integral $j$ separately. Hadronic states with integral $j$ have odd fermion number and correspond to baryons; states with half-integral $j$ have even fermion number and correspond to mesons.

First consider the heavy baryons. In this case, the "brown muck" is an object with spin-parity $j^P$ that can be represented by a totally symmetric, traceless tensor $A^{\mu_1 \cdots \mu_j}$ subject to the transversality condition $v_\mu \, A^{\mu_1 \cdots \mu_j} = 0$. States are said to have "natural" parity if $P = (-1)^j$, and "unnatural" parity otherwise. The composite heavy baryon can be represented by the tensor wave function

$$\psi^{\mu_1 \cdots \mu_j} = u_h \, A^{\mu_1 \cdots \mu_j} \, . \tag{179}$$

Under a connected Lorentz transformation $\Lambda$, this object transforms as a spinor-tensor field

$$\psi^{\mu_1 \cdots \mu_j} \to \Lambda^{\mu_1}_{\nu_1} \ldots \Lambda^{\mu_j}_{\nu_j} \, D(\Lambda) \, \psi^{\nu_1 \cdots \nu_j} \, , \tag{180}$$

where $D(\Lambda) = \exp(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu})$ is the usual spinor representation of $\Lambda$. A heavy quark spin rotation $\widetilde{\Lambda}$, on the other hand, acts only on $u_h$; hence

$$\psi^{\mu_1 \cdots \mu_j} \to D(\widetilde{\Lambda}) \, \psi^{\mu_1 \cdots \mu_j} \, . \tag{181}$$

Here $\widetilde{\Lambda}$ is restricted to spatial rotations (in the rest frame). The infinitesimal form of $D(\widetilde{\Lambda})$ was considered in (117). The simplest but important case $j^P = 0^+$ corresponds to the ground-state $\Lambda_Q$-baryon with total spin-parity $J^P = \frac{1}{2}^+$. It can be represented by a spinor $u_\Lambda$. Since the light degrees of freedom are in a configuration of total spin zero, the spin of the baryon is carried by the heavy quark, and the spinor $u_\Lambda$ coincides with the heavy quark spinor. Hence

$$\psi_\Lambda = u_\Lambda(v,s) = u_h(v,s) \, . \tag{182}$$

For $j \geq 0$, the object $\psi^{\mu_1 \cdots \mu_j}$ does not transform irreducibly under the Lorentz group, but is a linear combination of two components with total spin $j \pm \frac{1}{2}$. These correspond to a degenerate doublet of physical states, which only differ in the orientation of the heavy quark spin relative to the angular momentum of the light degrees of freedom. The nonrelativistic quark model suggests that one should identify the states with $j^P = 1^+$ with the $\Sigma_Q$ ($J^P = \frac{1}{2}^+$) and $\Sigma_Q^*$ ($J^P = \frac{3}{2}^+$) baryons. In the quark model, these states contain a heavy quark and a light vector diquark with no orbital angular momentum. Unlike the $\Lambda_Q$-baryons, they have unnatural parity. This implies that decays between $\Lambda_Q$ and $\Sigma_Q^{(*)}$ must be described by parity-odd form factors[206,207].

Next consider the heavy mesons. I shall only discuss the case $j^P = \frac{1}{2}^-$ in detail. Since quarks and antiquarks have opposite intrinsic parity, the corresponding physical states with "natural" parity are the ground-state pseudoscalar ($J^P = 0^-$) and vector ($J^P = 1^-$) mesons. As before, the heavy quark is represented by a spinor $u_h(v,s)$ subject to the condition $\not{v}\, u_h(v,s) = u_h(v,s)$. The light degrees of freedom as a whole transform under the Lorentz group as an antiquark moving at velocity $v$. They are described by an antifermion spinor $\bar{v}_\ell(v,s')$ satisfying $\bar{v}_\ell(v,s')\not{v} = -\bar{v}_\ell(v,s')$. The ground-state mesons can be represented by the composite object $\psi = u_h\,\bar{v}_\ell$, which is a $4\times 4$ Dirac matrix with two spinor indices, one for the heavy quark and one for the "brown muck". Under a connected Lorentz transformation $\Lambda$, the meson wave function $\psi$ transforms as

$$\psi \to D(\Lambda)\,\psi\,D^{-1}(\Lambda)\,, \tag{183}$$

whereas under a heavy quark spin rotation $\tilde{\Lambda}$

$$\psi \to D(\tilde{\Lambda})\,\psi\,. \tag{184}$$

The composite $\psi$ represents a linear combination of the physical pseudoscalar and vector meson states. It is easiest to identify these states in the rest frame, where $u_h$ has only upper components, whereas $\bar{v}_\ell$ has only lower components. The nonvanishing components of $\psi$ are thus contained in a $2\times 2$ matrix, which can be written as a linear combination of the identity $I$ and the Pauli matrices $\sigma^i$. Let me choose the quantization axis in 3-direction and work with the rest-frame spinor basis

$$u_h(\Uparrow) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\,,\quad u_h(\Downarrow) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\,,\quad v_\ell(\uparrow) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\,,\quad v_\ell(\downarrow) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\,. \tag{185}$$

Then a basis of states is:

$$\begin{aligned}
(\,\Uparrow\downarrow + \Downarrow\uparrow\,) &= -\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\,,\\
(\,\Uparrow\downarrow - \Downarrow\uparrow\,) &= -\begin{pmatrix}0 & \sigma^3\\ 0 & 0\end{pmatrix}\,,\\
\sqrt{2}\,(\,\Uparrow\uparrow\,) &= -\frac{1}{\sqrt{2}}\begin{pmatrix}0 & \sigma^1+i\sigma^2\\ 0 & 0\end{pmatrix}\,,\\
\sqrt{2}\,(\,\Downarrow\downarrow\,) &= -\frac{1}{\sqrt{2}}\begin{pmatrix}0 & \sigma^1-i\sigma^2\\ 0 & 0\end{pmatrix}\,.
\end{aligned} \tag{186}$$

Let me furthermore define two transverse polarization vectors $\epsilon_\pm$ and a longitudinal polarization vector $\epsilon_3$ by

$$\epsilon_\pm^\mu = \frac{1}{\sqrt{2}}\,(0,1,\pm i,0)\,,\qquad \epsilon_3^\mu = (0,0,0,1)\,. \tag{187}$$

It is then obvious to identify the pseudoscalar ($P$) and vector ($V$) meson states as[22,204,205]

$$\begin{aligned} P(\vec{v}=0) &= -\frac{1+\gamma^0}{2}\,\gamma_5\,, \\ V(\vec{v}=0,\epsilon) &= \quad \frac{1+\gamma^0}{2}\,\not{\epsilon}\,. \end{aligned} \tag{188}$$

The second state in (186) has longitudinal polarization, whereas the last two states have transverse polarization.

To get familiar with this representation, consider the action of the spin operator $\vec{\mathbf{\Sigma}}$ on $\psi$. A matrix representation of the components $\mathbf{\Sigma}^i$ in the rest frame is $\Sigma^i = \frac{1}{2}\gamma_5\gamma^0\gamma^i$, and the action of the operator $\mathbf{\Sigma}^i$ on the meson wave function is $\mathbf{\Sigma}^i\,\psi = [\Sigma^i,\psi]$. Using this, one finds

$$\begin{aligned} \vec{\mathbf{\Sigma}}^2 P &= \mathbf{\Sigma}^3 P = 0\,, \\ \vec{\mathbf{\Sigma}}^2\,V(\epsilon) &= 2\,V(\epsilon)\,, \\ \mathbf{\Sigma}^3\,V(\epsilon_\pm) &= \pm V(\epsilon_\pm)\,, \\ \mathbf{\Sigma}^3\,V(\epsilon_3) &= 0\,, \end{aligned} \tag{189}$$

which shows that $P$ has total spin zero, and $V$ has total spin one. Next consider the action of the heavy quark spin operator $\vec{\mathbf{S}}$. It has the same matrix representation as $\vec{\mathbf{\Sigma}}$, but only acts on the heavy quark spinor in $\psi$: $\mathbf{S}^i\,\psi = S^i\,\psi$, with $S^i = \Sigma^i$. It follows that

$$\begin{aligned} \mathbf{S}^3 P &= \frac{1}{2}\,V(\epsilon_3)\,, \\ \mathbf{S}^3\,V(\epsilon_3) &= \frac{1}{2}\,P\,, \\ \mathbf{S}^3\,V(\epsilon_\pm) &= \pm\frac{1}{2}\,V(\epsilon_\pm)\,, \end{aligned} \tag{190}$$

in accordance with the spin assignments for the heavy quark in (186).

In a general frame, the tensor wave functions in (188) can be readily generalized in a Lorentz-covariant way by replacing $\gamma^0$ with $\not{v}$. The covariant representation of states can be used to determine in a very efficient way the structure of hadronic matrix elements in the effective theory. The goal is to find a minimal form factor decomposition consistent with Lorentz covariance, parity invariance of the strong interactions, and heavy quark symmetry. The flavour symmetry is manifest when one uses mass-independent wave functions. The correct transformation properties under the spin symmetry are guaranteed when one collects a spin-doublet of states into a single object. In case of the ground-state pseudoscalar and vector mesons, for instance, one introduces a covariant tensor wave function $\mathcal{M}(v)$ that represents both $P(v)$ and $V(v,\epsilon)$ by

$$\mathcal{M}(v) = \frac{1+\not{v}}{2}\,\begin{cases} -\gamma_5\,; & \text{pseudoscalar meson,} \\ \not{\epsilon}\,; & \text{vector meson.} \end{cases} \tag{191}$$

It has the important property $\mathcal{M}(v) = P_+ \, \mathcal{M}(v) \, P_-$, where $P_\pm = \frac{1}{2}(1 \pm \not{v})$. This is often used to simplify expressions.

Consider now a transition between two heavy mesons, $M(v) \to M'(v')$, mediated by a renormalized effective current operator $\bar{h}'_{v'} \, \Gamma \, h_v$, which changes a heavy quark $Q$ into another heavy quark $Q'$. According to the Feynman rules of HQET, the "heavy quark part" of the decay amplitude is simply proportional to $\bar{u}'_h \, \Gamma \, u_h$; interactions of the heavy quarks with gluons do not modify the Dirac structure of $\Gamma$. Since the heavy quark spinors are part of the tensor wave functions associated with the hadron states, it follows that the amplitude must be proportional to $\overline{\mathcal{M}}'(v') \, \Gamma \, \mathcal{M}(v)$. This is a Dirac matrix with two indices representing the light degrees of freedom. Since the total matrix element is a scalar, these indices must be contracted with those of a matrix $\Xi$. Hence one may write

$$\langle M'(v') | \, \bar{h}'_{v'} \, \Gamma \, h_v \, | M(v) \rangle = \mathrm{Tr}\Big\{ \, \Xi(v, v', \mu) \, \overline{\mathcal{M}}'(v') \, \Gamma \, \mathcal{M}(v) \Big\} \, . \tag{192}$$

The matrix $\Xi$ contains all long-distance dynamics. It is a most complicated object, only constrained by the symmetries of the effective theory. Heavy quark symmetry requires that it be independent of the spins and masses of the heavy quarks, as well as of the Dirac structure of the current. Hence, $\Xi$ can only be a function of the meson velocities and of the renormalization scale $\mu$. Lorentz covariance and parity invariance imply that $\Xi$ must transform as a scalar with even parity. This allows the decomposition

$$\Xi(v, v', \mu) = \Xi_1 + \Xi_2 \, \not{v} + \Xi_3 \, \not{v}' + \Xi_4 \, \not{v} \, \not{v}' \, , \tag{193}$$

with coefficients $\Xi_i = \Xi_i(v \cdot v', \mu)$. But using the projection properties of the tensor wave functions, one finds that under the trace

$$\Xi(v, v', \mu) \to \Xi_1 - \Xi_2 - \Xi_3 + \Xi_4 \equiv -\xi(v \cdot v', \mu) \, . \tag{194}$$

Therefore[22],

$$\langle M'(v') | \, \bar{h}'_{v'} \, \Gamma \, h_v \, | M(v) \rangle = -\xi(v \cdot v', \mu) \, \mathrm{Tr}\Big\{ \, \overline{\mathcal{M}}'(v') \, \Gamma \, \mathcal{M}(v) \Big\} \, . \tag{195}$$

The sign is chosen such that the universal form factor $\xi(v \cdot v', \mu)$ coincides with the Isgur–Wise function, which is the single form factor that describes semileptonic weak decay processes of heavy mesons in the infinite quark mass limit. Equation (195) summarizes in a compact way the results derived in Section 3.4. Using the explicit form of the meson wave functions, one can readily recover the relations (128), (129), and (138). The only new feature is that the Isgur–Wise function depends on the renormalization scale $\mu$. This is necessary to compensate the scale dependence of the Wilson coefficients, which multiply the renormalized current operators in the short-distance expansion.

*3.9. Meson Decay Form Factors*

One of the most important applications of heavy quark symmetry is to derive relations between the form factors parametrizing the exclusive weak decays $B \to D\,\ell\,\bar\nu$ and $B \to D^*\ell\,\bar\nu$. A detailed theoretical understanding of these processes is necessary for a reliable determination of the element $|V_{cb}|$ of the Kobayashi–Maskawa matrix. Let me start by introducing a set of six hadronic form factors $h_i(w)$, which parametrize the relevant meson matrix elements of the flavour-changing vector and axial vector currents $V^\mu = \bar c\,\gamma^\mu b$ and $A^\mu = \bar c\,\gamma^\mu\gamma_5\,b$:

$$\begin{aligned}
\langle D(v')|\,V^\mu\,|B(v)\rangle &= h_+(w)\,(v+v')^\mu + h_-(w)\,(v-v')^\mu\,,\\
\langle D^*(v',\epsilon)|\,V^\mu\,|B(v)\rangle &= i\,h_V(w)\,\epsilon^{\mu\nu\alpha\beta}\,\epsilon^*_\nu\,v'_\alpha\,v_\beta\,,\\
\langle D^*(v',\epsilon)|\,A^\mu\,|B(v)\rangle &= h_{A_1}(w)\,(w+1)\,\epsilon^{*\mu} - \Big[h_{A_2}(w)\,v^\mu + h_{A_3}(w)\,v'^\mu\Big]\epsilon^*\cdot v\,.
\end{aligned} \tag{196}$$

Here $w = v\cdot v'$ is the velocity transfer of the mesons, and $\epsilon$ denotes the polarization vector of the $D^*$-meson. At leading order in the $1/m_Q$ expansion, one can derive expressions for these form factors by using the short-distance expansion (169) of the flavour-changing currents, as well as the tensor formalism outlined above. According to (177), the $\mu$ dependence of the Wilson coefficients of any bilinear heavy quark current can be factorized into a universal function $K_{\rm hh}(w,\mu)$, which is normalized at zero recoil. The $\mu$ dependence of this function has to cancel against that of the Isgur–Wise function. One can use this fact to define a renormalization-group invariant Isgur–Wise form factor as

$$\xi_{\rm ren}(w) \equiv \xi(w,\mu)\,K_{\rm hh}(w,\mu)\,, \qquad \xi_{\rm ren}(1) = 1\,. \tag{197}$$

Neglecting terms of order $1/m_Q$, one then obtains[197]

$$\langle M'(v')|\,J^\mu\,|M(v)\rangle = -\xi_{\rm ren}(w)\sum_{i=1}^3 \widehat C_i^{(5)}(w)\,{\rm Tr}\Big\{\,\overline{\mathcal M}'(v')\,\Gamma_i\,\mathcal M(v)\Big\}\,, \tag{198}$$

where the Dirac matrices $\Gamma_i$ have been defined in (170) and (171). It is now straightforward to evaluate the traces to find

$$\begin{aligned}
h_+(w) &= \left[\widehat C_1(w) + \frac{w+1}{2}\,\Big(\widehat C_2(w) + \widehat C_3(w)\Big)\right]\xi_{\rm ren}(w)\,,\\
h_-(w) &= \frac{w+1}{2}\,\Big[\widehat C_2(w) - \widehat C_3(w)\Big]\,\xi_{\rm ren}(w)\,,\\
h_V(w) &= \widehat C_1(w)\,\xi_{\rm ren}(w)\,,\\
h_{A_1}(w) &= \widehat C_1^5(w)\,\xi_{\rm ren}(w)\,,\\
h_{A_2}(w) &= \widehat C_2^5(w)\,\xi_{\rm ren}(w)\,,\\
h_{A_3}(w) &= \Big[\widehat C_1^5(w) + \widehat C_3^5(w)\Big]\,\xi_{\rm ren}(w)\,.
\end{aligned} \tag{199}$$

This is the correct generalization of (129) and (138) in the presence of short-distance corrections. The fact that, to leading order in $1/m_Q$, the meson form factors are given in terms of a single universal function $\xi_{\rm ren}(w)$ was the discovery of Isgur and Wise[13]. Short-distance QCD corrections affect the form factors in a calculable way. Their effects are contained in the various combinations of short-distance coefficients, which can be evaluated using the numerical results given in Table 1.

### *3.10. Power Corrections and Luke's Theorem*

Using the covariant tensor formalism and the short-distance expansions of the effective Lagrangian and currents beyond the leading order in $1/m_Q$, one can investigate in a systematic way the structure of power corrections to the relations derived in the previous section. I have given a simple (tree-level) example of the structure of power corrections in (143). For the more complicated case of meson weak decay form factors, the analysis at order $1/m_Q$ was performed by Luke[24]. Later, short-distance corrections were included to all orders in perturbation theory[203,208]. Falk and myself have also analysed the structure of $1/m_Q^2$ corrections for both meson and baryon weak decay form factors[187]. I shall not discuss these rather technical issues in detail, but nevertheless give you the main results.

Luke has shown that, for transitions between two heavy ground-state (pseudoscalar or vector) mesons, the $1/m_Q$ corrections can be parametrized by a set of four additional universal functions of the velocity transfer $w$. The most important outcome of his analysis concerns the zero recoil limit, where an analogue of the Ademollo–Gatto theorem[80] can be proved. This is Luke's theorem[24], which states that the matrix elements describing the leading $1/m_Q$ corrections to meson decay amplitudes vanish at zero recoil. As a consequence, in the limit $v = v'$ there are no terms of order $1/m_Q$ in the hadronic matrix elements in (196). This theorem is independent of the structure of the Wilson coefficients and thus valid to all orders in perturbation theory[187,208,209].

There is considerable confusion in the literature about the implications of this result. It is often claimed that the theorem would protect any meson decay rate, or even all form factors that are normalized in the spin-flavour symmetry limit, from first-order power corrections at zero recoil. However, these claims are erroneous. The reason is simple but somewhat subtle. Luke's theorem only protects the form factors $h_+$ and $h_{A_1}$ in (196), since all the others are multiplied by kinematic factors which vanish for $v = v'$. In fact, an explicit calculation shows that the $1/m_Q$ corrections to $h_-$, $h_V$, $h_{A_2}$, and $h_{A_3}$ do not vanish at zero recoil. The fact that these functions are kinematically suppressed does not imply that they could not contribute to physical decay rates. This is often overlooked. Consider, as an example, the process $B \to D\,\ell\,\bar\nu$ in the limit of vanishing lepton mass. By angular momentum conservation, the two pseudoscalar mesons must be in a relative $p$-wave in order to match the helicities of the lepton pair. The amplitude is proportional to the velocity $|\vec v_D|$ of the $D$-meson in the $B$-meson rest frame, which leads to a factor $(w^2-1)$ in

the decay rate. In such a situation, form factors that are kinematically suppressed can contribute[210]. Indeed, the $B \to D\,\ell\,\bar\nu$ decay rate is proportional to

$$(w^2-1)^{3/2}\left|h_+(w)-\frac{m_B-m_D}{m_B+m_D}\,h_-(w)\right|^2. \tag{200}$$

Both form factors, $h_+$ and $h_-$, contribute with similar strength to the rate near $w=1$, although $h_-$ is kinematically suppressed in (196). Consequently, the decay rate at zero recoil does receive corrections of order $1/m_Q$. The situation is different for $B \to D^*\ell\,\bar\nu$ transitions[125]. Because the vector meson has spin one, the decay can proceed in an $s$-wave, and there is no helicity suppression near zero recoil. One finds that close to $w=1$ the decay rate is proportional to $(w^2-1)^{1/2}\,|h_{A_1}(w)|^2$. Since the form factor $h_{A_1}$ is protected by Luke's theorem, these transitions are ideally suited for a precision measurement of $|\,V_{cb}|$ from an extrapolation of the momentum spectrum of the $D^*$-meson to zero recoil[12,125]. There, the normalization of the decay rate is known in a model-independent way up to corrections of order $1/m_Q^2$:

$$h_{A_1}(1)=\eta_A+\delta_{1/m^2}+\dots\,, \tag{201}$$

where $\eta_A \simeq 0.99$ contains the short-distance corrections [cf. (178)].

One expects higher-order power corrections to be of order $\delta_{1/m^2} \sim (\Lambda_{\rm QCD}/m_c)^2 \sim 3\%$, but of course such a naïve estimate could be too optimistic. For a precision measurement of $|\,V_{cb}|$, it is important to know the structure of such corrections in more detail. Although in principle straightforward, the analysis of $1/m_Q^2$ corrections in HQET is a tedious enterprise[187]. Three classes of corrections have to be distinguished: matrix elements of local dimension-five current operators, "mixed" corrections resulting from the combination of corrections to the current and to the Lagrangian, and corrections from one or two insertions of operators from the effective Lagrangian into matrix elements of the leading-order currents. Within these classes, one can distinguish corrections proportional to $1/m_b^2$, $1/m_c^2$, or $1/m_b m_c$. More than thirty new universal functions are necessary to parametrize the second-order power corrections to meson form factors. When radiative corrections are neglected, eleven combinations of these functions contribute to the hadronic form factors $h_i(w)$. The general results greatly simplify at zero recoil, however. There the equation of motion and the Ward identities of the effective theory can be used to prove that matrix elements of local dimension-five current operators, as well as matrix elements of time-ordered products containing a dimension-four current and an insertion from the effective Lagrangian, can be expressed in terms of only two parameters $\lambda_1$ and $\lambda_2$, which are related to the $1/m_Q$ corrections to the physical meson masses. Moreover, the conservation of the flavour-diagonal vector current in the full theory forces certain combinations of the universal functions to vanish at zero recoil. The consequence is that whenever a form factor is protected by Luke's theorem, the structure of second-order power corrections at zero recoil becomes rather simple. The quantity $\delta_{1/m^2}$ in (201) is of this type. A careful estimate based on some mild model assumptions gives[187,211] $-3\% < \delta_{1/m^2} < -1\%$. One can

combine this result with (201) to obtain one of the most important, and certainly most precise predictions of HQET:

$$h_{A_1}(1) = 0.97 \pm 0.04\,. \tag{202}$$

### *3.11. Properties of the Isgur–Wise Function*

The establishment of heavy quark symmetry as an exact limit of the strong interactions enables one to derive approximate relations between decay amplitudes, and normalization conditions for certain form factors, which are similar to the relations and normalization conditions that can be derived for Goldstone-boson scattering amplitudes from the low-energy theorems of current algebra. HQET provides the framework for a systematic investigation of the corrections to the limit of an exact spin-flavour symmetry. The output of such a model-independent analysis is a short-distance expansion of decay amplitudes, in which the dependence on the heavy quark masses is explicit. At each order in the $1/m_Q$ expansion, the long-distance physics is parametrized by a minimal set of universal form factors, which are independent of the heavy quark masses. As presented so far, the analysis is completely model-independent. Since hadronic decay processes are of a genuine nonperturbative nature, however, it is clear that predictions that can be made based on symmetries only are limited. In particular, very little can be said on general grounds about the properties of the form factors of the effective theory. But there is a lot of information contained in these functions. Much like the hadron structure functions probed in deep-inelastic scattering, they are fundamental nonperturbative quantities in QCD, which describe the properties of the light degrees of freedom in the background of the colour field provided by the heavy quarks. Since a static colour source is the most direct way to probe the strong interactions of quarks and gluons at large distances, a theoretical understanding of the universal form factors would not only enlarge the predictive power of HQET, but would also teach us in a very direct way about the nonperturbative nature of the strong interactions.

The leading-order Isgur–Wise function $\xi(w)$ plays a central role in the description of the weak decays of heavy mesons. It contains the long-distance physics associated with the strong interactions of the light degrees of freedom and cannot be calculated from first principles. Nevertheless, some important properties of this function can be derived on general grounds, such as its normalization at zero recoil, which is a consequence of current conservation. According to (128), the Isgur–Wise function is the elastic form factor of a ground-state heavy meson in the limit where power corrections are negligible. As such, $\xi(w)$ must be a monotonically decreasing function of the velocity transfer $w = v \cdot v'$, which is analytic in the cut $w$-plane with a branch point at $w = -1$, corresponding to the threshold $q^2 = 4m_Q^2$ for heavy quark pair production. However, being obtained from a limiting procedure, the Isgur–Wise function can have stronger singularities than the physical elastic form factor. In fact, the short-distance corrections contained in the function $K_{\rm hh}(w,\mu)$

lead to an essential singularity at $w = -1$ in the renormalized Isgur–Wise function defined in (197).

When using a phenomenological parametrization of the universal form factor, one should incorporate the above properties. Some legitimate forms suggested in the literature are:

$$\begin{aligned}
\xi_{\rm BSW}(w) &= \frac{2}{w+1}\,\exp\left\{-(2\varrho^2-1)\,\frac{w-1}{w+1}\right\},\\
\xi_{\rm ISGW}(w) &= \exp\left\{-\varrho^2\,(w-1)\right\},\\
\xi_{\rm pole}(w) &= \left(\frac{2}{w+1}\right)^{2\varrho^2}.
\end{aligned} \tag{203}$$

The first function is the form factor derived by Rieckert and myself[210] from an analysis of the BSW model[81], the second one corresponds to the ISGW model[83], and the third one is a pole-type ansatz. Of particular interest is the behaviour of the Isgur–Wise function close to zero recoil, which is determined by the slope parameter $\varrho^2 > 0$ defined by $\xi'(1) = -\varrho^2$, so that

$$\xi(w) = 1 - \varrho^2\,(w-1) + O[(w-1)^2]\,. \tag{204}$$

It is important to realize that the kinematic region accessible in semileptonic decays is small ($1 < w < 1.6$). As long as $\varrho^2$ is the same, different functional forms of $\xi(w)$ will give similar results. A precise knowledge of the slope parameter would thus basically determine the Isgur–Wise function in the physical region.

Bjorken has shown that $\varrho^2$ is related to the form factors of transitions of a ground-state heavy meson into excited states, in which the light degrees of freedom carry quantum numbers $j^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$, by a sum rule which is an expression of quark–hadron duality: In the infinite mass limit, the inclusive sum of the probabilities for decays into hadronic states is equal to the probability for the free quark transition. If one normalizes the latter probability to unity, the sum rule has the form[204,212,213]

$$\begin{aligned}
1 = {}& \frac{w+1}{2}\left\{|\xi(w)|^2 + \sum_l |\xi^{(l)}(w)|^2\right\}\\
&+ (w-1)\left\{2\sum_m |\tau_{1/2}^{(m)}(w)|^2 + (w+1)^2\sum_n |\tau_{3/2}^{(n)}(w)|^2\right\}\\
&+ O[(w-1)^2]\,,
\end{aligned} \tag{205}$$

where $l, m, n$ label the radial excitations of states with the same spin-parity quantum numbers. The sums are understood in a generalized sense as sums over discrete states and integrals over continuum states. The terms on the right-hand side of the sum rule in the first line correspond to transitions into states with "brown muck" quantum numbers $j^P = \frac{1}{2}^-$. The ground state gives a contribution proportional to

the Isgur–Wise function, and excited states contribute proportionally to analogous functions $\xi^{(l)}(w)$. Because at zero recoil these states must be orthogonal to the ground state, it follows that $\xi^{(l)}(1) = 0$, and one can conclude that the corresponding contributions to (205) are of order $(w-1)^2$. The contributions in the second line correspond to transitions into states with $j^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$. Because of the change in parity, these are $p$-wave transitions. The amplitudes are proportional to the velocity $|\vec{v}_f| = (w^2-1)^{1/2}$ of the final state in the rest frame of the initial state, which explains the suppression factor $(w-1)$ in the decay probabilities. The functions $\tau_j(w)$ are the analogues of the Isgur–Wise function for these transitions. Transitions into excited states with quantum numbers other than the above proceed via higher partial waves and are suppressed by at least a factor $(w-1)^2$.

For $w = 1$, eq. (205) reduces to the normalization condition for the Isgur–Wise function. The Bjorken sum rule is obtained by expanding in powers of $(w-1)$ and keeping terms of first order. Taking into account the definition of the slope parameter $\varrho^2$ in (204), one finds that[204,212]

$$\varrho^2 = \frac{1}{4} + \sum_m |\tau_{1/2}^{(m)}(w)|^2 + 2\sum_n |\tau_{3/2}^{(n)}(w)|^2 > \frac{1}{4}\,. \tag{206}$$

Notice that the lower bound is due to the prefactor $\frac{1}{2}(w+1)$ of the first term in (205) and is of purely kinematic origin. In the analogous sum rule for $\Lambda_Q$-baryons, this factor is absent, and consequently the slope parameter of the baryon Isgur–Wise function is only subject to the trivial constraint[210,214] $\varrho^2 > 0$.

Based on various model calculations, there is a general belief that the contributions of excited states in the Bjorken sum rule are sizeable, and that $\varrho^2$ is substantially larger than 1/4. For instance, Blok and Shifman have estimated the contributions of the lowest-lying excited states to (206) using QCD sum rules and find that[215] $0.35 < \varrho^2 < 1.15$. The experimental observation that semileptonic $B$-decays into excited $D^{**}$-mesons have a large branching ratio of about[216] 2.5% gives further support to the importance of such contributions.

Voloshin has derived another sum rule involving the form factors for transitions into excited states, which is the analogue of the "optical sum rule" for the dipole scattering of light in atomic physics. It reads[217]

$$\frac{m_M - m_Q}{2} = \sum_m E_{1/2}^{(m)}\,|\tau_{1/2}^{(m)}(w)|^2 + 2\sum_n E_{3/2}^{(n)}\,|\tau_{3/2}^{(n)}(w)|^2\,, \tag{207}$$

where $E_j$ are the excitation energies relative to the mass $m_M$ of the ground-state heavy meson. The important point is that one can combine this relation with the Bjorken sum rule to obtain an upper bound for the slope parameter $\varrho^2$:

$$\varrho^2 < \frac{1}{4} + \frac{m_M - m_Q}{2E_{\rm min}}\,, \tag{208}$$

where $E_{\rm min}$ denotes the minimum excitation energy. In the quark model, one expects[§] that $E_{\rm min} \simeq m_M - m_Q$, and one may use this as an estimate to obtain $\varrho^2 < 0.75$.

[§]Strictly speaking, the lowest excited "state" contributing to the sum rule is $D + \pi$, which has an

The above discussion of the sum rules ignores renormalization effects. However, both the slope parameter $\varrho^2$ in (206) and the heavy quark mass $m_Q$ in (207) are renormalization-scheme dependent quantities. Although there exist some qualitative ideas of how to account for the $\mu$ dependence of the various parameters[212,217], there is currently no known way to include renormalization effects quantitatively. One should therefore consider the bounds on $\varrho^2$ as somewhat uncertain. To account for this, I relax the upper bound derived from the Voloshin sum rule and conclude that

$$0.25 < \varrho^2 < 1.0\,, \tag{209}$$

where it is expected that the actual value is close to the upper bound. Recently, de Rafael and Taron claimed to have derived an upper bound $\varrho^2 < 0.48$ from general analyticity arguments[218]. If true, this had severely constrained the form of the Isgur–Wise function near zero recoil. It took quite some time to become clear what went wrong with their derivation: The effects of resonances below the threshold for heavy meson pair production invalidate the argument[219–222]. It is possible to derive a new bound, which takes into account the known properties of the $\Upsilon$-states[223]. However, it is too loose to be of any phenomenological relevance, and one is thus left with the sum rule result (209).

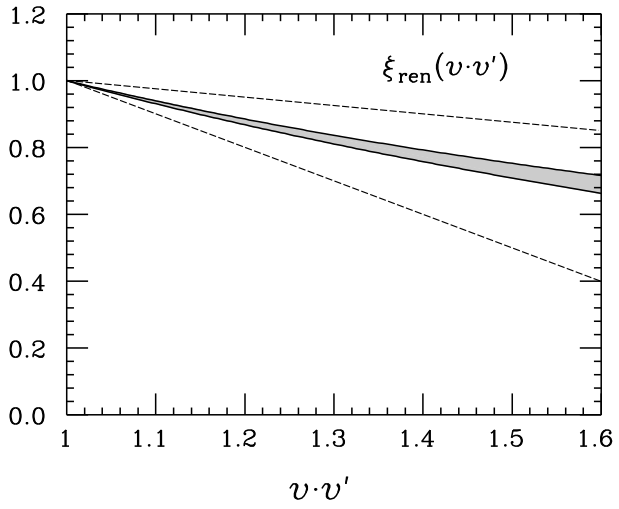


Figure 16: QCD sum rule prediction for the Isgur–Wise function in the kinematic region accessible in semileptonic decays[93]. The dashed lines indicate the bounds on the slope at $v \cdot v' = 1$ derived by Bjorken and Voloshin [see (209)].

Any sensible calculation of the Isgur–Wise function has to respect these bounds. As an example, I show in Fig. 16 the result of a QCD sum rule analysis of the

excitation energy spectrum with a threshold at $m_\pi$. However, one expects that this spectrum is broad, so that this contribution will not invalidate the upper bound for $\varrho^2$ derived here.

renormalized universal form factor[93–95]. Similar predictions have been obtained for the functions that describe the $1/m_Q$ corrections to the meson form factors[96].

### *3.12. Model-Independent Determination of* $|V_{cb}|$

One of the most important results of HQET is the prediction (202) of the normalization of the hadronic form factor $h_{A_1}$ at zero recoil. It can be used to obtain a model-independent measurement of the element $|V_{cb}|$ of the Kobayashi–Maskawa matrix. The semileptonic decay $B \to D^* \ell\, \bar\nu$ is ideally suited for this purpose[125]. Experimentally, this is a particularly clean mode, since the reconstruction of the $D^*$-meson mass provides a powerful rejection against background. From the theoretical point of view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$. In terms of the hadronic form factors defined in (196), one finds

$$\lim_{w\to 1} \frac{1}{\sqrt{w^2-1}}\, \frac{{\rm d}\Gamma(B\to D^*\ell\,\bar\nu)}{{\rm d}w} = \frac{G_F^2\,|V_{cb}|^2}{4\pi^3}\,(m_B - m_{D^*})^2\, m_{D^*}^3\, |h_{A_1}(1)|^2\,. \tag{210}$$

At zero recoil, the normalization of $h_{A_1}$ is known in a model-independent way with an accuracy of 4%. Ideally, then, one can extract $|V_{cb}|$ with a theoretical uncertainty of well below 10% from an extrapolation of the spectrum to $w = 1$.

Presently, the proposal to measure $|V_{cb}|$ close to zero recoil poses quite a challenge to the experimentalists. First, there is the fact that the decay rate vanishes at zero recoil because of phase space. Therefore the statistics gets worse as one tries to measure close to $w = 1$. However, I do not believe that this will be an important limitation of the method. The phase-space suppression is proportional to $\sqrt{w^2-1}$ and is in fact a rather mild one. When going from the endpoint $w_{\rm max} \simeq 1.5$ down to $w = 1.05$, the change in the statistical error in $|V_{cb}|^2$ due to the variation of the phase-space factor is not even a factor of 2. A more serious problem is related to the fact that, for experiments working on the $\Upsilon(4s)$ resonance, the zero-recoil limit corresponds to a situation where both the $B$- and the $D^*$-mesons are approximately at rest in the laboratory. Then the pion in the subsequent decay $D^* \to D\,\pi$ is very soft and can hardly be detected. Thus, the present experiments have to make cuts which disfavour the zero recoil region, leading to large systematic uncertainties for values of $w$ smaller than about 1.15. This second problem would be absent at an asymmetric $B$-factory, where the rest frame of the parent $B$-meson is boosted relative to the laboratory frame.

In view of these difficulties, one presently has to rely on an extrapolation over a wide range in $w$ to obtain a measurement of $|V_{cb}|$. In general, the differential decay rate can be written as[25,125]

$$\begin{aligned}\frac{{\rm d}\Gamma(B\to D^*\ell\,\bar\nu)}{{\rm d}w} &= \frac{G_F^2}{48\pi^3}\,(m_B - m_{D^*})^2\, m_{D^*}^3\, \eta_A^2\, \sqrt{w^2-1}\,(w+1)^2 \\ &\quad\times\left[1 + \frac{4w}{w+1}\,\frac{1-2wr+r^2}{(1-r)^2}\right] |V_{cb}|^2\, \widehat{\xi}^{\,2}(w)\,,\end{aligned} \tag{211}$$

where $\eta_A = 0.99$ is the short-distance correction to the form factor $h_{A_1}(w)$ at zero recoil, and $r = m_{D^*}/m_B$. Equation (211) is written in such a way that the deviations from the heavy quark symmetry limit are absorbed into the form factor $\widehat{\xi}(w)$, which in the absence of symmetry-breaking corrections would be the Isgur–Wise function. Since everything except $|V_{cb}|$ and $\widehat{\xi}(w)$ is known, a measurement of the differential decay rate is equivalent to a measurement of the product $|V_{cb}|\,\widehat{\xi}(w)$. However, theory predicts the normalization of $\widehat{\xi}(w)$ at zero recoil:

$$\widehat{\xi}(1) = \eta_A^{-1}\, h_{A_1}(1) = 1 + \delta_{1/m^2} = 0.98 \pm 0.04\,, \tag{212}$$

where the uncertainty comes from power corrections of order $1/m_Q^2$. Using this information, $|V_{cb}|$ and $\widehat{\xi}(w)$ can be obtained separately from a measurement of the differential decay rate.

I have applied this strategy for the first time[125] to the combined sample[224] of the data on $B^0 \to D^{*+}\ell\,\bar\nu$ decays collected until 1989 by the ARGUS and CLEO collaborations. For the extrapolation to zero recoil, I used the parametrizations given in (203) for the function $\widehat{\xi}(w)$, treating its slope at zero recoil as a free parameter. The result obtained for $|V_{cb}|$ was

$$|V_{cb}| \left(\frac{\tau_{B^0}}{1.5\ \text{ps}}\right)^{1/2} = 0.039 \pm 0.006\,. \tag{213}$$

Since this original analysis the data have changed. In particular, the branching ratio for $D^{*+} \to D^0\,\pi^+$ has increased from[52] 55% to[225] 68%. This lowers the decay rate for $B^0 \to D^{*+}\ell\,\bar\nu$, and correspondingly decreases $|V_{cb}|$ by 10%. However, the new data recently reported by the ARGUS[216] and CLEO[126] collaborations give a larger branching ratio than the old data, indicating that further changes in the analysis must have taken place. It is thus not possible to simply rescale the result for $|V_{cb}|$ given above.

In Fig. 17, I show the new ARGUS data[216] for the product $|V_{cb}|\,\widehat{\xi}(w)$. From an unconstrained fit using again the parametrizations in (203), the following value is obtained:

$$|V_{cb}| \left(\frac{\tau_{B^0}}{1.5\ \text{ps}}\right)^{1/2} = 0.049 \pm 0.008\,. \tag{214}$$

However, the fit gives very large values for the slope parameter $\varrho^2$, between 1.9 and 2.3. Since the slope of $\widehat{\xi}(w)$ agrees with the slope of the Isgur–Wise function up to power corrections of order $\Lambda_{\rm QCD}/m_c$, I believe that such large values cannot be tolerated in view of the Voloshin sum rule. In Fig. 17, I therefore show a fit to the data which uses the pole ansatz in (203) with the constraint that $\varrho^2 \le 1$. This leads to a significantly smaller value:

$$|V_{cb}| \left(\frac{\tau_{B^0}}{1.5\ \text{ps}}\right)^{1/2} = 0.037 \pm 0.006\,. \tag{215}$$

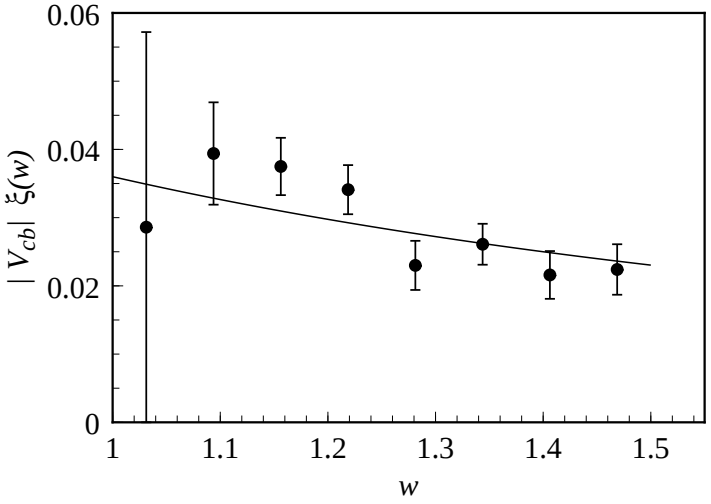


Figure 17: ARGUS data[216] for the product $|V_{cb}|\,\widehat{\xi}(w)$ as a function of the recoil $w = v \cdot v'$, assuming $\tau_{B^0} = 1.5$ ps. Using the normalization condition (212), $|V_{cb}|$ is obtained from an extrapolation to $w = 1$. The fit curve is explained in the text.

Very recently, the CLEO collaboration has reported new results for the recoil spectrum with higher statistics[126]. They have applied tight cuts in order to reduce the systematic errors. These data are shown in Fig. 18. The curves correspond to the results of a fit using the various functions in (203). Without any constraint on the slope at zero recoil, one obtains

$$|V_{cb}| \left(\frac{\tau_{B^0}}{1.5\ \text{ps}}\right)^{1/2} = 0.039 \pm 0.006\,, \tag{216}$$

together with values $1.0 \pm 0.4 < \varrho^2 < 1.2 \pm 0.7$. It is reassuring that the result for the slope parameter is in agreement with the bound derived from the Voloshin sum rule. Because of the careful analysis of systematic uncertainties, I consider these results to be the best numbers for $|V_{cb}|$ and $\varrho^2$ that are currently available.

The model-independent determination of $|V_{cb}|$ is one of the many applications of heavy quark symmetry that have been explored over the last years. I hope that the above presentation gives you a flavour of the potential of the HQET formalism. For a more comprehensive discussion, I refer to my review paper[25].

### *3.13. Exercises*

- Go through the derivation of the effective Lagrangian of HQET and prove eqs. (104)–(113).

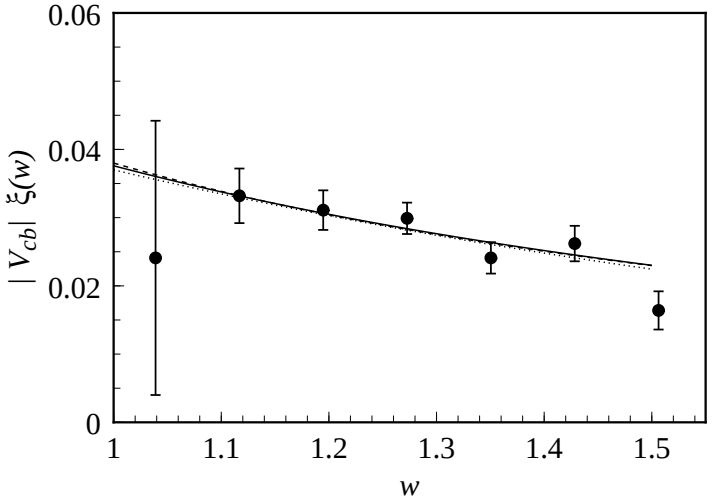


Figure 18: CLEO data[126] for the product $|V_{cb}|\,\widehat{\xi}(w)$ as a function of the recoil $w$.

- Derive the Feynman rules depicted in Fig. 12 from the effective Lagrangian (114).
- Convince yourself that (138) is the correct covariant expression for the pseudoscalar-to-vector transition matrix element.
- Derive the one-loop expression given in (149) for the wave-function renormalization constant $Z_h$.
- By performing the appropriate Dirac traces in (198), derive the expressions given in (199) for the meson form factors $h_i(w)$.
- Derive the Bjorken sum rule (206) by taking the limit $w \to 1$ in (205).

## Concluding Remarks

In these lectures, I have presented an introduction to the theory and phenomenology of heavy quark masses and mixing angles, as well as to the exciting new field of heavy quark spin-flavour symmetry. The synthesis of these topics is interesting in itself, in that it combines rather mature subjects with one of the most active fields of research in particle physics. As far as quark masses and mixing angles are concerned, the theoretical concepts have been set many years ago. Yet new theoretical tools have to be developed to determine these parameters more and more accurately. Heavy quark effective field theory, on the other hand, has received broad attention in recent years only, although some ideas of a spin-flavour symmetry for heavy quarks had been around for much longer. These new developments start to have an impact on the precision with which one can determine some elements of the Kobayashi–Maskawa matrix. I have discussed this in detail for the extraction of $|V_{cb}|$ from the semileptonic decays $B \to D^* \ell\, \bar\nu$.

Besides reviewing the status of the theoretical developments, my purpose was to convince you that $B$-physics is a rich and diverse area of research. Presently, this field is characterized by a very fruitful interplay between theory and experiments, which has led to many significant discoveries and developments on both sides. Heavy quark physics has the potential to determine important parameters of the flavour sector of the electroweak theory, and at the same time provides stringent tests of the standard model. On the other hand, weak decays of hadrons containing a heavy quark are an ideal laboratory to study the nature of nonperturbative phenomena in QCD.

When I presented these lectures in the summer of 1993, the future of $B$-physics was uncertain. Today, the prospects for further significant developments in this field look rather promising. With the approval of the first asymmetric $B$-factory at SLAC, and with ongoing $B$-physics programs at the existing facilities at Cornell, Fermilab, and LEP, there are clearly $B$eautiful times ahead of us!

*Acknowledgements:* It is a great pleasure to thank the organizers of TASI-93 for the invitation to present these lectures and for providing a stimulating and relaxing atmosphere, which helped to initiate many physics discussions. Coming to TASI-93 also gave me the opportunity to meet, for the first time, my collaborator Zoltan Ligeti, and to complete a piece of work with him. As far as fun and sports are concerned, I will remember an honest attempt to climb Longs Peak with Zoltan, and a most pleasant day of rock climbing in the Boulder area with Tom DeGrand. Writing lecture notes, on the other hand, is a painful and time-consuming experience. It is only because of the continuous support and queries of Stuart Raby that I finally finished this manuscript. I very much appreciated his patience with me.

Many discussions with colleagues helped me to prepare the lectures and had an impact on the form of these notes. I am particularly grateful to my collaborators Adam Falk, Zoltan Ligeti, Yossi Nir, Michael Luke, and Thomas Mannel. During

the school, I have enjoyed discussions with many of the students, as well as with Andy Cohen, Chris Hill, and Peter Lepage. Before coming to Boulder, talking to Michael Peskin was an invaluable asset in preparing the lectures. In addition, I would like to thank David Cassel and Persis Drell for keeping me updated about the CLEO analysis of semileptonic $B$-decays. Finally, it is my great pleasure to thank Suzy Vascotto for her careful reading of the manuscript, and Maria Girone for teaching me `PAW`.

## References


1. H. Albrecht et al. (ARGUS collaboration), Phys. Lett. B **192**, 245 (1987).
2. C. Albajar et al. (UA1 collaboration), Phys. Lett. B **186**, 247 (1987).
3. H. Albrecht et al. (ARGUS collaboration), Phys. Lett. B **234**, 409 (1990).
4. R. Fulton et al. (CLEO collaboration), Phys. Rev. Lett. **64**, 16 (1990).
5. H. Albrecht et al. (ARGUS collaboration), Phys. Lett. B **255**, 297 (1991).
6. J. Bartelt et al. (CLEO collaboration), Phys. Rev. Lett. **71**, 4111 (1993).
7. R. Ammar et al. (CLEO collaboration), Phys. Rev. Lett. **71**, 674 (1993).
8. E.V. Shuryak, Phys. Lett. B **93**, 134 (1980); Nucl. Phys. B **198**, 83 (1982).
9. J.E. Paschalis and G.J. Gounaris, Nucl. Phys. B **222**, 473 (1983); F.E. Close, G.J. Gounaris, and J.E. Paschalis, Phys. Lett. B **149**, 209 (1984).
10. S. Nussinov and W. Wetzel, Phys. Rev. D **36**, 130 (1987).
11. M.B. Voloshin and M.A. Shifman, Yad. Fiz. **45**, 463 (1987) [Sov. J. Nucl. Phys. **45**, 292 (1987)].
12. M.B. Voloshin and M.A. Shifman, Yad. Fiz. **47**, 801 (1988) [Sov. J. Nucl. Phys. **47**, 511 (1988)].
13. N. Isgur and M.B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).
14. E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981).
15. W.E. Caswell and G.P. Lepage, Phys. Lett. B **167**, 437 (1986).
16. E. Eichten, in: Field Theory on the Lattice, edited by A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) **4**, 170 (1988).
17. G.P. Lepage and B.A. Thacker, in: Field Theory on the Lattice, edited by A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) **4**, 199 (1988).
18. H.D. Politzer and M.B. Wise, Phys. Lett. B **206**, 681 (1988); **208**, 504 (1988).
19. E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990); **243**, 427 (1990).
20. B. Grinstein, Nucl. Phys. B **339**, 253 (1990).
21. H. Georgi, Phys. Lett. B **240**, 447 (1990).
22. A.F. Falk, H. Georgi, B. Grinstein, and M.B. Wise, Nucl. Phys. B **343**, 1 (1990).
23. A.F. Falk, B. Grinstein, and M.E. Luke, Nucl. Phys. B **357**, 185 (1991).
24. M.E. Luke, Phys. Lett. B **252**, 447 (1990).
25. M. Neubert, SLAC preprint SLAC-PUB-6263 (1993), to appear in Phys. Rep.
26. R. Tarrach, Nucl. Phys. B **183**, 384 (1981).
27. W.A. Bardeen, A.J. Buras, D.W. Duke, and T. Muta, Phys. Rev. D **18**, 3998 (1978); A.J. Buras, Rev. Mod. Phys. **52**, 199 (1980).
28. G. 't Hooft, Nucl. Phys. B **61**, 455 (1973).
29. M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B **147**, 385 (1979); **147**, 448 (1979).
30. A.F. Falk, M. Neubert, and M.E. Luke, Nucl. Phys. B **388**, 363 (1992).
31. M. Neubert, Phys. Rev. D **46**, 3914 (1992).
32. G. 't Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).
33. C.G. Bollini and J.J. Giambiagi, Phys. Lett. B **40**, 566 (1972); Nuovo Cim.

B **12**, 20 (1972).
34. D.J. Gross, in: Methods in Field Theory, edited by C.R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976), p. 141.
35. N. Gray, D.J. Broadhurst, W. Grafe, and K. Schilcher, Z. Phys. C **48**, 673 (1990).
36. O.V. Tarasov, JINR preprint P2-82-900 (1982), unpublished.
37. S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. B **135**, 457 (1984).
38. D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).
39. H.D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).
40. W.E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).
41. D.R.T. Jones, Nucl. Phys. B **75**, 531 (1974).
42. A.A. Belavin and A.A. Migdal, JETP Lett. **19**, 181 (1974) [ZhETF Pis. Red. **19**, 317 (1974)].
43. E.G. Floratos, D.A. Ross, and C.T. Sachrajda, Nucl. Phys. B **129**, 66 (1977) [E: **139**, 545 (1978)].
44. A.J. Buras, M. Jamin, M.E. Lautenbacher, and P.H. Weisz, Nucl. Phys. B **370**, 69 (1992).
45. T. Appelquist and J. Carazzone, Phys. Rev. D **11**, 2856 (1975).
46. B. Pendleton and G.G. Ross, Phys. Lett. B **98**, 291 (1981).
47. C.T. Hill, Phys. Rev. D **24**, 691 (1981); C.T. Hill, C.N. Leung, and S. Rao, Nucl. Phys. B **262**, 517 (1985).
48. J. Bagger, S. Dimopoulos, and E. Massó, Phys. Rev. Lett. **55**, 920 (1985).
49. M. Carena et al., Nucl. Phys. B **369**, 33 (1992); M. Carena, S. Pokorski, and C.E.M. Wagner, Nucl. Phys. B **406**, 59 (1993).
50. V. Barger, M.S. Berger, and P. Ohmann, Phys. Rev. D **47**, 1093 (1993).
51. W.A. Bardeen, C.T. Hill, and M. Lindner, Phys. Rev. D **41**, 1647 (1990).
52. K. Hikasa et al., Review of Particle Properties, Phys. Rev. D **45**, Part II (1992).
53. B. Guberina, R. Meckbach, R.D. Peccei, and R. Rückl, Nucl. Phys. B **184**, 476 (1981).
54. L.J. Reinders, Phys. Rev. D **38**, 947 (1988); L.J. Reinders, H.R. Rubinstein, and S. Yazaki, Phys. Rep. **127**, 1 (1985).
55. V.A. Novikov et al., Phys. Rep. **41**, 1 (1978).
56. M.B. Voloshin, Yad. Fiz. **29**, 1368 (1979) [Sov. J. Nucl. Phys. **29**, 703 (1979)].
57. M.B. Voloshin and Yu.M. Zaitsev, Usp. Fiz. Nauk **152**, 361 (1987) [Sov. Phys. Usp. **30**, 553 (1987)].
58. S. Narison, Phys. Lett. B **210**, 238 (1988); **308**, 305 (1993).
59. E. Bagan, P. Ball, V.M. Braun, and H.G. Dosch, Phys. Lett. B **278**, 457 (1992).
60. M. Neubert, Phys. Rev. D **46**, 1076 (1992).
61. I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, CERN preprint CERN-TH.7171/94 (1994).
62. M. Beneke and V.M. Braun, Munich preprint MPI-PhT/94-9 (1994).
63. G. 't Hooft, in: The Whys of Subnuclear Physics, Proceedings of the 15th

International School on Subnuclear Physics, Erice, Sicily, 1977, edited by A. Zichichi (Plenum Press, New York, 1979), p. 943.
64. B. Lautrup, Phys. Lett. B **69**, 438 (1977).
65. G. Parisi, Phys. Lett. B **76**, 65 (1978); Nucl. Phys. B **150**, 163 (1979).
66. F. David, Nucl. Phys. B **234**, 237 (1984); **263**, 637 (1986).
67. A.H. Mueller, Nucl. Phys. B **250**, 327 (1985); Phys. Lett. B **308**, 355 (1993).
68. A.J. Buras and M.K. Harlander, in: Heavy Flavours, edited by A.J. Buras and M. Lindner (World Scientific, Singapore,1992), p. 58.
69. Y. Nir, in: The Third Family and the Physics of Flavour, Proceedings of the 20th SLAC Summer Institute on Particle Physics, Stanford, California, 1992, edited by L. Vassilian (SLAC Report No. 412, Stanford, 1993), p. 81; Y. Nir and H.R. Quinn, Annu. Rev. Nucl. Part. Sci. **42**, 221 (1992).
70. N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
71. M. Kobayashi and K. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
72. C. Jarlskog, in: CP Violation, edited by C. Jarlskog (World Scientific, Singapore, 1989), p. 3.
73. C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Z. Phys. C **29**, 491 (1985).
74. L.L. Chau and W.-Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984).
75. L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).
76. A. Sirlin and R. Zucchini, Phys. Rev. Lett. **57**, 1994 (1986); A. Sirlin, Phys. Rev. D **35**, 3423 (1987).
77. W. Jaus and G. Rasche, Phys. Rev. D **35**, 3420 (1987).
78. H. Leutwyler and M. Roos, Z. Phys. C **25**, 91 (1984).
79. J.F. Donoghue, B.R. Holstein, and S.W. Klimt, Phys. Rev. D **35**, 934 (1987).
80. M. Ademollo and R. Gatto, Phys. Rev. Lett. **13**, 264 (1964).
81. M. Wirbel, B. Stech and M. Bauer, Z. Phys. C **29**, 637 (1985); M. Bauer, B. Stech, and M. Wirbel, Z. Phys. C **34**, 103 (1987).
82. B. Grinstein, M.B. Wise, and N. Isgur, Phys. Rev. Lett. **56**, 298 (1986).
83. N. Isgur, D. Scora, B. Grinstein, and M.B. Wise, Phys. Rev. D **39**, 799 (1989).
84. J.G. Körner and G.A. Schuler, Z. Phys. C **38**, 511 (1988) [E: **41**, 690 (1989)]; **46**, 93 (1990).
85. S.J. Brodsky and G.P. Lepage, Phys. Rev. D **22**, 2157 (1980).
86. M. Suzuki, Nucl. Phys. B **258**, 553 (1985).
87. T. Altomari and L. Wolfenstein, Phys. Rev. Lett. **58**, 1583 (1987).
88. T.M. Aliev and V.L. Eletskij, Yad. Fiz. **38**, 1537 (1983) [Sov. J. Nucl. Phys. **38**, 936 (1983)]; T.M. Aliev, V.L. Eletskij, and Y.I. Kogan, Yad. Fiz. **40**, 823 (1984) [Sov. J. Nucl. Phys. **40**, 527 (1984)].
89. A.A. Ovchinnikov, V.A. Slobodenyuk, Z. Phys. C **44**, 433 (1989); A.A. Ovchinnikov, Yad. Fiz. **50**, 1433 (1989) [Sov. J. Nucl. Phys. **50**, 891 (1989)].
90. V.N. Baier and A.G. Grozin, Z. Phys. C **47**, 669 (1990).
91. P. Ball, V.M. Braun, H.G. Dosch, and M. Neubert, Phys. Lett. B **259**, 481 (1991); P. Ball, V.M. Braun, and H.G. Dosch, Phys. Rev. D **44**, 3567 (1991);

Phys. Lett. B **273**, 316 (1991).
92. P. Ball, Phys. Lett. B **281**, 133 (1992); in: Perturbative QCD and Hadronic Interactions, Proceedings of the 27th Rencontres de Moriond, edited by J. Tran Thanh Van, Gif-sur-Yvette, France, 1992 (Editions Frontières, Gif-sur-Yvette, 1992), p. 343.
93. M. Neubert, Phys. Rev. D **45**, 2451 (1992); **47**, 4063 (1993).
94. M. Neubert, V. Rieckert, B. Stech, and Q.P. Xu, in: Heavy Flavours, edited by A.J. Buras and M. Lindner (World Scientific, Singapore, 1992), p. 286.
95. A.V. Radyushkin, Phys. Lett. B **271**, 218 (1991).
96. M. Neubert, Z. Ligeti, and Y. Nir, Phys. Lett. B **301**, 101 (1993); Phys. Rev. D **47**, 5060 (1993); **49**, 1302 (1994).
97. J.H. Christenson, J.W. Cronin, V.L. Fitch, and R. Turlay, Phys. Rev. Lett. **13**, 138 (1964); Phys. Rev. B **140**, 74 (1965).
98. N. Isgur and M.B. Wise, Phys. Rev. D **42**, 2388 (1990).
99. C.O. Dib and F. Vera, Phys. Rev. D **47**, 3938 (1993).
100. G. Burdman, Z. Ligeti, M. Neubert, and Y. Nir, Phys. Rev. D **49**, 2331 (1994).
101. A. Ali and E. Pietarinen, Nucl. Phys. B **154**, 519 (1979).
102. G. Corbò, Nucl. Phys. B **212**, 99 (1983); N. Cabibbo, G. Corbò, and L. Maiani, Nucl. Phys. B **155**, 93 (1979).
103. M. Jezabek and J.H. Kühn, Nucl. Phys. B **320**, 20 (1989).
104. G. Altarelli et al., Nucl. Phys. B **208**, 365 (1982).
105. A. Bareiss and E.A. Paschos, Nucl. Phys. B **327**, 353 (1989); C.H. Jin, W.F. Palmer, and E.A. Paschos, Dortmund preprint DO-TH-93/21 (1993).
106. M. Neubert, Phys. Rev. D **49**, 3392 (1994).
107. M. Neubert, CERN preprint CERN-TH.7113/93 (1993), to appear in Phys. Rev. D **49**, no. 9 (1994).
108. I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, CERN preprint CERN-TH.7129/93 (1993).
109. A.F. Falk, E. Jenkins, A.V. Manohar, and M.B. Wise, San Diego preprint UCSD/PTH-93-38 (1993).
110. T. Mannel and M. Neubert, CERN preprint CERN-TH.7156/94 (1994).
111. J. Chay, H. Georgi, and B. Grinstein, Phys. Lett. B **247**, 399 (1990).
112. I.I. Bigi, N.G. Uraltsev, and A.I. Vainshtein, Phys. Lett. B **293**, 430 (1992); I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, Phys. Rev. Lett. **71**, 496 (1993); I.I. Bigi et al., in: Proceedings of the Annual Meeting of the Division of Particles and Fields of the American Physical Society, Batavia, Illinois, 1992, edited by C. Albright et al. (World Scientific, Singapore, 1993), p. 610.
113. B. Blok, L. Koyrakh, M.A. Shifman, and A.I. Vainshtein, Phys. Rev. D **49**, 3356 (1994).
114. A.V. Manohar and M.B. Wise, Phys. Rev. D **49**, 1310 (1994).
115. A.F. Falk, M. Luke, and M.J. Savage, Phys. Rev. D **49**, 3367 (1994).
116. T. Mannel, Nucl. Phys. B **413**, 396 (1994).

117. L. Koyrakh, Minnesota preprint TPI-MINN-93/47-T (1993).
118. A.F. Falk, Z. Ligeti, M. Neubert, and Y. Nir, CERN preprint CERN-TH.7124/93 (1993), to appear in Phys. Lett. B.
119. S. Balk, J.G. Körner, D. Pirjol, and K. Schilcher, Mainz preprint MZ-TH/93-32 (1993).
120. M. Luke and M.J. Savage, Phys. Lett. B **321**, 88 (1994).
121. G.P. Yost et al., Review of Particle Properties, Phys. Lett. B **204** (1988).
122. S. Stone, to appear in the 2nd edition of: *B* Decays, edited by S. Stone (World Scientific, Singapore, 1991).
123. M. Danilov, to appear in: Proceedings of the International Europhysics Conference on High Energy Physics, Marseille, France, July 1993, edited by J. Carr and M. Perrottet (Editions Frontières, Gif-sur-Yvettes).
124. T. Hessing, to appear in: Electroweak Interactions and Unified Theories, Proceedings of the 29th Rencontres de Moriond, Méribel, France, March 1993.
125. M. Neubert, Phys. Lett. B **264**, 455 (1991).
126. G. Crawford et al. (CLEO collaboration), to appear in: Proceedings of the 16th International Symposium on Lepton and Photon Interaction, Ithaca, New York, Aug. 1993.
127. J.D. Bjorken, presented in discussions at the Workshop on Experiments, Detectors and Experimental Areas for the Supercollider, Berkeley, California, July 1987 (Proceedings edited by R. Donaldson and M.G.D. Gilchriese, World Scientific, Singapore, 1988).
128. C. Jarlskog and R. Stora, Phys. Lett. B **208**, 268 (1988).
129. I.I. Bigi, V.A. Khoze, N.G. Uraltsev, and A.I. Sanda, in: CP Violation, edited by C. Jarlskog (World Scientific, Singapore, 1989), p. 175.
130. C.O. Dib, I. Dunietz, F.J. Gilman, and Y. Nir, Phys. Rev. D **41**, 1522 (1990); F.J. Gilman and Y. Nir, Annu. Rev. Nucl. Part. Sci. **40**, 213 (1990).
131. C.S. Kim, J.L. Rosner, and S.-P. Yuan, Phys. Rev. D **42**, 96 (1990).
132. M. Schmidtler and K.R. Schubert, Z. Phys. C **53**, 347 (1992).
133. M. Lusignoli, L. Maiani, G. Martinelli, and L. Reina, Nucl. Phys. B **369**, 139 (1992).
134. G.R. Harris and J.L. Rosner, Phys. Rev. D **45**, 946 (1992).
135. H. Albrecht et al. (ARGUS collaboration), Z. Phys. C **55**, 357 (1992).
136. J. Bartelt et al. (CLEO collaboration), Phys. Rev. Lett. **71**, 1680 (1993).
137. D. Abbaneo, to appear in: Electroweak Interactions and Unified Theories, Proceedings of the 29th Rencontres de Moriond, Méribel, France, March 1993.
138. A.J. Buras, M. Jamin, and P.H. Weisz, Nucl. Phys. B **347**, 491 (1990).
139. T. Inami and C.S. Lim, Progr. Theor. Phys. **65**, 297 (1981); **65**, 1772 (1981).
140. A.J. Buras, Phys. Rev. Lett. **46**, 1354 (1981).
141. G. Buchalla, A.J. Buras, and M.K. Harlander, Nucl. Phys. B **349**, 1 (1991).
142. J.F. Donoghue, E. Golowich, and B.R. Holstein, Phys. Lett. B **119**, 412 (1982).

143. A. Pich and E. de Rafael, Phys. Lett. B **158**, 477 (1985); Nucl. Phys. B **358**, 311 (1991).
144. R. Decker, in: Hadronic Matrix Elements and Weak Decays, edited by A.J. Buras, J.-M. Gérard, and W. Huber, Nucl. Phys. B (Proc. Suppl.) **7a**, 180 (1989), and references therein.
145. W.A. Bardeen, A.J. Buras, and J.-M. Gérard, Phys. Lett. B **211**, 343 (1988).
146. M.B. Gavela et al., Phys. Lett. B **206**, 113 (1988); Nucl. Phys. B **306**, 677 (1988).
147. G.W. Kilcup, S.R. Sharpe, R. Gupta, and A. Patel, Phys. Rev. Lett. **64**, 25 (1990).
148. C. Bernard and A. Soni, in: Lattice 89, edited by N. Cabibbo et al., Nucl. Phys. B (Proc. Suppl.) **17**, 495 (1990).
149. C. Sachrajda, in: QCD-20 Years Later, Proceedings of the Workshop on QCD, Aachen, Germany, 1992, edited by P.M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993), p. 668, and references therein.
150. S.R. Sharpe and A. Patel, Washington preprint UW-PT-93-1 (1993); S.R. Sharpe, to appear in: Proceedings of the 11th Symposium on Lattice Field Theory (Lattice 93), Dallas, Texas, Oct. 1993.
151. H. Burckhardt et al. (NA31 collaboration), Phys. Lett. B **206**, 169 (1988).
152. J.R. Patterson et al. (E731 collaboration), Phys. Rev. Lett. **64**, 1491 (1990).
153. G. Buchalla, A.J. Buras, and M.K. Harlander, Nucl. Phys. B **337**, 313 (1990).
154. A.J. Buras, M. Jamin, and M.E. Lautenbacher, Nucl. Phys. B **408**, 209 (1993).
155. M. Ciuchini, E. Franco, G. Martinelli, and L. Reina, Phys. Lett. B **301**, 263 (1993); Rome preprint 93/913 (1993).
156. S. Bertolini, in: Higgs Particle(s): Physics Issues and Searches in High-Energy Collisions, Proceedings of the 8th INFN Eloisatron Project Workshop, Erice, Sicily, 1989, edited by A. Ali (Plenum Press, New York, 1990), p. 243.
157. A. Ali, to appear in the 2nd edition of: $B$ Decays, edited by S. Stone (World Scientific, Singapore, 1991).
158. S. Bertolini, F. Borzumati, and A. Masiero, Phys. Rev. Lett. **59**, 180 (1987).
159. N.G. Deshpande et al., Phys. Rev. Lett. **59**, 183 (1987).
160. B. Grinstein, R. Springer, and M.B. Wise, Phys. Lett. B **202**, 138 (1988); Nucl. Phys. B **339**, 269 (1990).
161. R. Grigjanis, P.J. O'Donnel, M. Sutherland, and H. Navelet, Phys. Lett. B **213**, 355 (1988); **223**, 239 (1989); **237**, 252 (1990).
162. G. Cella, G. Curci, G. Ricciardi, and A. Vicere, Phys. Lett. B **248**, 181 (1990).
163. M. Misiak, Phys. Lett. B **269**, 161 (1991); Nucl. Phys. B **393**, 23 (1993).
164. M. Ciuchini et al., Phys. Lett. B **316**, 127 (1993).
165. A. Ali and C. Greub, Z. Phys. C **49**, 431 (1991); Phys. Lett. B **259**, 182 (1991); **287**, 191 (1992).
166. A. Ali and T. Mannel, Phys. Lett. B **264**, 447 (1991) [E: **274**, 526 (1992)].

167. P. Ball, Munich preprint TUM-T31-43/93 (1993).
168. S. Narison, CERN preprint CERN-TH.7166/94 (1994).
169. H. Georgi, in: Perspectives in the Standard Model, Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics (TASI-91), Boulder, Colorado, 1991, edited by R.K. Ellis, C.T. Hill, and J.D. Lykken (World Scientific, Singapore, 1992), p. 589.
170. B. Grinstein, in: High Energy Phenomenology, Proceedings of the Workshop on High Energy Phenomenology, Mexico City, Mexico, 1991, edited by M.A. Péres and R. Huerta (World Scientific, Singapore, 1992), p. 161.
171. N. Isgur and M.B. Wise, in: Heavy Flavours, edited by A.J. Buras and M. Lindner (World Scientific, Singapore, 1992), p. 234.
172. T. Mannel, in: QCD-20 Years Later, Proceedings of the Workshop on QCD, Aachen, Germany, 1992, edited by P.M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993), p. 634.
173. T. Appelquist and H.D. Politzer, Phys. Rev. Lett. **34**, 43 (1975).
174. M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B **120**, 316 (1977).
175. E. Witten, Nucl. Phys. B **122**, 109 (1977).
176. J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
177. K. Wilson, Phys. Rev. **179**, 1499 (1969); Phys. Rev. D **3**, 1818 (1971).
178. W. Zimmermann, Ann. of Phys. **77**, 536 (1973); **77**, 570 (1973).
179. G. Altarelli and L. Maiani, Phys. Lett. B **52**, 351 (1974).
180. M.K. Gaillard and B.W. Lee, Phys. Rev. Lett. **33**, 108 (1974).
181. F.J. Gilman and M.B. Wise, Phys. Rev. D **27**, 1128 (1983).
182. T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. B **368**, 204 (1992).
183. J. Soto and R. Tzani, Phys. Lett. B **297**, 358 (1992).
184. H. Georgi, B. Grinstein, and M.B. Wise, Phys. Lett. B **252**, 456 (1990).
185. N. Isgur and M.B. Wise, Phys. Rev. Lett. **66**, 1130 (1991).
186. D. Buskulic et al. (ALEPH collaboration), Phys. Lett. B **311**, 425 (1993).
187. A.F. Falk and M. Neubert, Phys. Rev. D **47**, 2965 (1993); **47**, 2982 (1993).
188. G. Preparata and W.I. Weisberger, Phys. Rev. **175**, 1965 (1968).
189. X. Ji and M.J. Musolf, Phys. Lett. B **257**, 409 (1991).
190. D.J. Broadhurst and A.G. Grozin, Phys. Lett. B **267**, 105 (1991).
191. D.J. Broadhurst, N. Gray, and K. Schilcher, Z. Phys. C **52**, 111 (1991).
192. B.S. DeWitt, Phys. Rev. **162**, 1195 (1967).
193. G. 't Hooft, in: Functional and Probabilistic Methods in Quantum Field Theory, Proceedings of the 12th Winter School of Theoretical Physics, Karpacz, Poland, Acta Univ. Wratisl. **38**, Vol. 1 (1975).
194. D. Boulware, Phys. Rev. D **23**, 389 (1981).
195. L.F. Abbott, Nucl. Phys. B **185**, 189 (1981); Acta Phys. Pol. B **13**, 33 (1982).
196. H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D **12**, 3159 (1975).
197. M. Neubert, Phys. Rev. D **46**, 2212 (1992).
198. G.P. Korchemsky and A.V. Radyushkin, Nucl. Phys. B **283**, 342 (1987); Phys. Lett. B **279**, 359 (1992); G.P. Korchemsky, Mod. Phys. Lett. A **4**,

1257 (1989).
199. A.F. Falk and B. Grinstein, Phys. Lett. B **247**, 406 (1990); **249**, 314 (1990).
200. M. Neubert, Nucl. Phys. B **371**, 149 (1992).
201. W. Kilian, P. Manakos, and T. Mannel, Phys. Rev. D **48**, 1321 (1993).
202. M. Luke and A.V. Manohar, Phys. Lett. B **286**, 348 (1992).
203. M. Neubert, Phys. Lett. B **306**, 357 (1993); Phys. Rev. D **49**, 1542 (1994).
204. J.D. Bjorken, in: Results and Perspectives in Particle Physics, Proceedings of the 4th Rencontres de Physique de la Vallé d'Aoste, La Thuile, Italy, 1990, edited by M. Greco (Editions Frontières, Gif-sur-Yvette, 1990), p. 583; in: Gauge Bosons and Heavy Quarks, Proceedings of the 18th SLAC Summer Institute on Particle Physics, Stanford, California, 1990, edited by J.F. Hawthorne (SLAC Report No. 378, Stanford, 1991), p. 167.
205. A.F. Falk, Nucl. Phys. B **378**, 79 (1992).
206. H.D. Politzer, Phys. Lett. B **250**, 128 (1990).
207. T. Mannel, W. Roberts, and Z. Ryzak, Phys. Lett. B **271**, 421 (1991).
208. M. Neubert, Nucl. Phys. B **416**, 786 (1994).
209. P. Cho and B. Grinstein, Phys. Lett. B **285**, 153 (1992).
210. M. Neubert and V. Rieckert, Nucl. Phys. B **382**, 97 (1992).
211. T. Mannel, CERN preprint CERN-TH.7162/94 (1994).
212. N. Isgur and M.B. Wise, Phys. Rev. D **43**, 819 (1991).
213. J.D. Bjorken, I. Dunietz, and J. Taron, Nucl. Phys. B **371**, 111 (1992).
214. N. Isgur, M.B. Wise, and M. Youssefmir, Phys. Lett. B **254**, 215 (1991).
215. B. Blok and M. Shifman, Phys. Rev. D **47**, 2949 (1993).
216. H. Albrecht et al. (ARGUS collaboration), Z. Phys. C **57**, 533 (1993).
217. M.B. Voloshin, Phys. Rev. D **46**, 3062 (1992).
218. E. de Rafael and J. Taron, Phys. Lett. B **282**, 215 (1992).
219. C. Dominguez, J.G. Körner, and D. Pirjol, Phys. Lett. B **301**, 257 (1993).
220. A.F. Falk, M. Luke, and M.B. Wise, Phys. Lett. B **299**, 123 (1993).
221. B. Grinstein and P.F. Mende, Phys. Lett. B **299**, 127 (1993).
222. C.E. Carlson et al., Phys. Lett. B **299**, 133 (1993).
223. E. de Rafael and J. Taron, Marseille preprint CPT-93/P.2908 (1993).
224. D. Bortoletto and S. Stone, Phys. Rev. Lett. **65**, 2951 (1990).
225. S. Stone, in: Heavy Flavours, edited by A.J. Buras and M. Lindner (World Scientific, Singapore, 1992), p. 334.